논리연구 24-3(2021) pp. 245-280

The Identity of Proofs and the Criterion for Admissible Reductions*

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【Abstract】Dag Prawitz (1971) put forward the idea that an admissible reduction process does not affect the identity of proofs represented by derivations in natural deduction. The idea relies on his conjecture that two derivations represent the same proof if and only if they are equivalent in the sense that they are reflexive, transitive and symmetric closure of the immediate reducibility relation.

Schroeder-Heister and Tranchini (2017) accept Prawitz's conjecture and propose the triviality test as the criterion for admissible reductions. In the present paper, we will consider two main troubles of the triviality test. The first is the obscurity of a method of evaluating admissible reductions. The second is the circularity problem that the triviality test already assumes the set of admissible reduction procedures. For the solution of the problems, we will propose the spoiler test which immunes the problems of the triviality test and has the role of the criterion for admissible reductions. At last, we shall cover a plausible problem of the spoiler test that can be caused by Crabbé's case.

【Key Words】Admissible reductions, Ekman paradox, The identity of proofs, Crabbé's case, The triviality test.

Received: Jul. 13th, 2021. Revised: Oct. 3rd, 2021. Accepted: Oct. 7th, 2021.

* This research has been supported by the AMOREPACIFIC Foundation. An early version of this paper was presented at the United Congress of Corean Philosophers (CCUP) in October 2020. I am grateful to Hanjong So, Byungdeok Lee, Ilho Park, and Inkyo Chung for their thoughtful suggestions and comments on the earlier version of this paper. I would like to thank anonymous referees for their helpful comments and encouragement.

1 Introduction

Many mathematical theorems have more than one proof. For example, Euclid's theorem which asserts that there are infinitely many prime numbers has at least three versions, such as Euclid's, Euler's, and Erdös's proof. Similarly, there is more than one formalization of the same proof in proof theory. Proof-theorists have investigated which derivation is a *canonical representation* of the proofs represented in the system considered as the way that the numerals are canonical notations for natural numbers.

Dag Prawitz (1971, Sec. I.3.5.6) has proposed the idea that normal derivations serve the special role of canonical representations. Normal derivations can be obtained by certain reduction procedures. Any derivations which can be a normal derivation with the finite steps of reductions are called normalizable derivations. A normalizable derivation shares the special role of a canonical representation of its normal derivation. Prawitz explained in terms of the equivalence class that normal and normalizable derivations represent the same proofs. He considered the equivalence relation between derivations that is reflexive, transitive and symmetric closure of the immediate reducibility relation between derivations. Then, he conjectured that two derivations represent the same proof if and only if they are equivalent. He thought that the right-to-left direction is relatively unproblematic and so claimed that an admissible reduction does not affect the identity of a proof represented. Hence, an admissible reduction preserves the identity between proofs. In this perspective, the admissible reductions can serve as the criterion for the identity of proofs.

On the other hand, assuming that we intuitively recognize which formal derivations represent the same informal proof, Schroeder-Heister and Tranchini (2017) accept Prawitz's conjecture and suggest the criterion for admissible reduction procedures. The idea is to evaluate whether, after a new reduction is added in a set of reductions for an intended system, it can be shown that two derivations of the same conclusion previously belonging to two distinct equivalence classes still belong to the distinct equivalence classes. If it can, they say that the new reduction does not trivialize the identity of proofs; otherwise, it *trivializes the identity of proofs*. A newly added reduction procedure is *admissible* if it does not trivialize the identity of proofs; otherwise, it is inadmissible. We call their evaluation method for admissible reductions, the *triviality test*.

The present paper aims to point out the problems of the triviality test and to propose an improved approach. In Section 2, we shall introduce Ekman's paradox and the triviality test. We will see how the triviality test blocks Ekman's reduction. Section 3 put forward two main problems of the triviality test: the obscurity of a test method and the circularity problem. With the solutions to the problems, we will introduce the spoiler test as the refined criterion for admissible reduction procedures in Section 4. The spoiler test can restrict the application of the Ekman reduction process.

Schroeder-Heister and Tranchini (2018) have considered their triviality test has a defect that it cannot solve the problem of Tennant's proof-theoretic criterion for paradoxicality occurred by Crabbé's case. Since every reduction process of Crabbé's case passes the spoiler test, the spoiler test may be seen as having the same defect. Section 5 argues that not every problem with Tennant's criterion is raised by inadmissible reductions. When we distinguish between the problems of Tennant's criterion and of inadmissible reductions, the spoiler test need not solve every problem of Tennant's criterion but it only has the role of evaluating admissible reductions.

2 Ekman's Paradox and the Triviality Test

Neil Tennant (1982) has proposed a proof-theoretic criterion for paradoxicality that a genuine paradox is a derivation of an unacceptable conclusion which employs a certain form of *id est* inferences and generates an infinite reduction sequence. Schroeder-Heister and Tranchini (2017) have suggested the triviality test to solve the problem of Tennant's criterion. They have considered Ekman's paradox taken from Jan Ekman (1998) to show that Tennant's criterion overgenerates in the sense that there exists a derivation which is intuitively non-paradoxical but satisfies the criterion. To solve the problem of overgeneration, Tennant (2016, 2017) has refined his criterion and suggested an additional condition that all elimination rules stated in generalized form. Unfortunately, Choi (2019) and Schroeder-Heister and Tranchini (2018) argued that Tennant's refined version of the criterion overgenerates again by showing that there is a generalized version of Ekman's paradox satisfying the refined criterion.

Schroeder-Heister and Tranchini (2017) diagnosed that Ekman's paradox applies too loose reduction procedure, called the *Ekman reduction*. Since they focused on the issue of which reduction process is an admissible one, they suggested the triviality test as the criterion for the admissibility of reduction procedures in order to restrict inadmissible reductions.

In this section, we set aside the issue of Tennant's criterion for paradoxicality and center on the triviality test. After we have preliminary notions and rules in Section 2.1, we shall see the triviality test and how it blocks the Ekman reduction process in Section 2.2.

2.1 Preliminaries: some terminologies and natural deduction rules

To begin with, we will introduce some terminologies and natural deduction rules. Our language has constants and quantifiers, $\wedge$, $\rightarrow$, $\perp$, and $\neg$ for conjunction, implication, absurdity, and negation respectively. We use φ, ψ, and σ for arbitrary formula. Let $\mathfrak{D}$ be a derivation of a given natural deduction system, used in the same manner as 'deduction' in Prawitz (1965). Also, we use the following conventions: if a derivation $\mathfrak{D}$ ends with a formula φ, we write as shown on the left below and φ is called an 'end-formula.' If it depends on a formula ψ, we write as shown on the right below.

$$\begin{array}{c} \mathfrak{D} \\ \varphi \end{array} \qquad \begin{array}{c} \psi \\ \mathfrak{D} \\ \varphi. \end{array}$$

Then, we have rules for $\wedge$ and $\rightarrow$ in the natural deduction style proposed by Prawitz (1965).

$$\dfrac{\begin{array}{cc} \mathfrak{D}_1 & \mathfrak{D}_2 \\ \varphi_1 & \varphi_2 \end{array}}{\varphi_1 \wedge \varphi_2} \wedge I \qquad \dfrac{\varphi_1 \wedge \varphi_2}{\varphi_i} \wedge E_{(i=1,2)} \qquad \dfrac{\begin{array}{c} [\varphi]^1 \\ \mathfrak{D}_1 \\ \psi \end{array}}{\varphi \rightarrow \psi} \rightarrow I_{,1} \qquad \dfrac{\begin{array}{cc} & \mathfrak{D}_2 \\ \varphi \rightarrow \psi & \varphi \end{array}}{\psi} \rightarrow E$$

$\neg\varphi$ is defined by $\varphi \rightarrow \perp$. We call the formulas directly above the line in each rule, 'premise,' and the formula directly below the line, 'conclusion.' *Assumptions* which can be discharged are in the square brackets, e.g. $[\varphi]$. When the same formula φ is introduced as assumptions with different superindexes, they are different types and not discharged simultaneously by a single rule. For instance, the following derivations have the same formula φ as their assumptions, yet their assumptions are discharged at different places.

$$\dfrac{\dfrac{[\varphi]^1 \quad [\varphi]^1}{\varphi \wedge \varphi}\wedge I}{\varphi \to (\varphi \wedge \varphi)}\to I_{,1} \qquad \dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi]^2}{\varphi \wedge \varphi}\wedge I}{\varphi \to (\varphi \wedge \varphi)}\to I_{,2}}{\varphi \to (\varphi \to (\varphi \wedge \varphi))}\to I_{,1}$$

The *open assumptions* of a derivation are the assumptions on which the end-formula depends. A derivation is called *closed* if it contains no open assumptions, otherwise it is called *open*. A *major premise* of the elimination rule for a constant is the premise containing the constant in the elimination rule and all other premises are *minor premises*. The *maximum formula* is the conclusion of an introduction rule at the same time the major premise of an elimination rule.

We accept the standard reduction procedures of Prawitz (1965, pp. 36-38). Let us consider any two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ having the same end-formula. We say that a derivation $\mathfrak{D}'$ is an *immediate subderivation* of $\mathfrak{D}_1$ if $\mathfrak{D}'$ is an initial part of $\mathfrak{D}_1$ ending with a premise of the last inference step in $\mathfrak{D}_1$. Let $\mathfrak{D}_1 \rhd \mathfrak{D}_2$ mean that $\mathfrak{D}_1$ *reduces* to $\mathfrak{D}_2$ by applying a single reduction step to an immediate subderivation $\mathfrak{D}'$ of $\mathfrak{D}_1$. For our convenience's sake, we will introduce standard reduction procedures for $\wedge$ and $\to$ as follows:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2 \\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}\wedge I}{\varphi_i}\wedge E_{(i=1,2)} \quad \rhd_\wedge \quad \begin{matrix}\mathfrak{D}_i \\ \varphi_i\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}[\varphi]^1 \\ \mathfrak{D}_1 \\ \psi\end{matrix}}{\varphi \to \psi}\to I_{,1} \quad \begin{matrix}\mathfrak{D}_2 \\ \varphi\end{matrix}}{\psi}\to E \quad \rhd_\to \quad \begin{matrix}\mathfrak{D}_2 \\ \varphi \\ \mathfrak{D}_1 \\ \psi\end{matrix}$$

The main role of these standard reduction procedures is to eliminate the maximum formula. When a derivation has no maximum formula, we say that it is in *normal form*. Let $\mathbb{R}$ be a set of reduction procedures. A reduction procedure $\begin{array}{c}\varphi_1,\dots,\varphi_n\\ \mathfrak{D}\\ \psi\end{array} \rhd \begin{array}{c}\varphi_1,\dots,\varphi_n\\ \mathfrak{D}'\\ \psi\end{array}$ in $\mathbb{R}$ is *closure under substitution* iff, for any derivation $\begin{array}{c}\mathfrak{D}_1 \\ \varphi_1\end{array},\dots,\begin{array}{c}\mathfrak{D}_n \\ \varphi_n\end{array}$, a reduction procedure $\begin{array}{c}\mathfrak{D}_1 \quad \mathfrak{D}_n\\ \varphi_1,\dots,\varphi_n\\ \mathfrak{D}\\ \psi\end{array} \rhd \begin{array}{c}\mathfrak{D}_1 \quad \mathfrak{D}_n\\ \varphi_1,\dots,\varphi_n\\ \mathfrak{D}'\\ \psi\end{array}$ is in $\mathbb{R}$ as well. Every reduction procedure in $\mathbb{R}$ is to be closed under substitution of derivations for open assumptions, and the notions of ‘normal’ and ‘normalizable’ are defined in the following ways[1]

Definition 2.1. A sequence $< \mathfrak{D}_1,\dots,\mathfrak{D}_i,\mathfrak{D}_{i+1},\dots >$ of derivations is a *reduction sequence* relative to $\mathbb{R}$ iff $\mathfrak{D}_i \rhd \mathfrak{D}_{i+1}$ relative to $\mathbb{R}$ where $1 \leqslant i$ for any natural number i. A derivation $\mathfrak{D}_1$ is *reducible* to $\mathfrak{D}_i$ ($\mathfrak{D}_1 \succ \mathfrak{D}_i$) relative to $\mathbb{R}$ iff there is a sequence $< \mathfrak{D}_1,\mathfrak{D}_2,\dots,\mathfrak{D}_i >$ relative to $\mathbb{R}$ where for each $j < i$, $\mathfrak{D}_j \rhd \mathfrak{D}_{j+1}$; $\mathfrak{D}_1$ is *irreducible* relative to $\mathbb{R}$ iff there is no derivation $\mathfrak{D}'$ to which $\mathfrak{D}_1 \rhd \mathfrak{D}'$ relative to $\mathbb{R}$ except $\mathfrak{D}_1$ itself.

Definition 2.2. A derivation $\mathfrak{D}$ is *normal* (or in *normal form*) relative to $\mathbb{R}$ iff $\mathfrak{D}$ is irreducible relative to $\mathbb{R}$, i.e. $\mathfrak{D}$ has no maximum formula. A reduction sequence *terminates* iff it has a finite number of derivations and its last derivation is in normal form. A derivation

[1] In Definition 2.1, for any term x and y, let $x \leqslant y$ mean that x is less than or equal to y. For Definition 2.2, for our convenience’s sake, we drop the ‘relative to $\mathbb{R}$’ in the suggested notions if there is no misunderstanding.

$\mathfrak{D}$ is *normalizable* relative to $\mathbb{R}$ iff there is a terminating reduction sequence relative to $\mathbb{R}$ starting from $\mathfrak{D}$.

In the next section, we shall introduce Schroeder-Heister and Tranchini's triviality test.

2.2 The triviality test and the Ekman reduction procedure

Schroeder-Heister and Tranchini (2017) have suggested the triviality test in order to restrict inadmissible reductions. Their target was the following form of reduction procedure introduced by Ekman (1998):

$$\dfrac{[\psi \to \varphi] \qquad \dfrac{[\varphi \to \psi] \quad \begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}}{\psi}\to E}{\varphi}\to E \qquad\qquad \trianglerighteq_E \qquad\qquad \begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}$$

We call $\trianglerighteq_E$ the *Ekman reduction* process. Ekman (1998) considered ψ in the last $\to E$−rule to be a similar kind of maximum formula. We call ψ in the last $\to E$−rule an *Ekman maximum formula*.

The triviality test is based on the thesis of Prawitz (1971, p. 257) that an admissible reduction should not affect the identity of proofs represented by derivations in the same equivalence class. He conjectures that two derivations represent the same proof if and only if they are equivalent. Prawitz's equivalence relation $\sim$ is defined via the reducibility relation $\succ$. We borrow the notion of the equivalence relation between derivations from Prawitz (1971, p. 255). Let $\mathbb{R}$ be any set of reduction procedures.[2]

[2]Although Prawitz (1971) considers a set of standard reduction procedures for rules of first-order intuitionistic and classical logic, since we focus on the test of an admissible reduction procedure in general, we consider arbitrary set of reduction procedures.

Definition 2.3. A derivation $\mathfrak{D}_1$ is *equivalent* to $\mathfrak{D}_i$ ($\mathfrak{D}_1 \sim \mathfrak{D}_i$) relative to $\mathbb{R}$ iff $\mathfrak{D}_1 \succ \mathfrak{D}_i$ or $\mathfrak{D}_i \succ \mathfrak{D}_1$ where $1 \leqslant i$ for any natural number i; otherwise, they are not equivalent ($\mathfrak{D}_1 \nsim \mathfrak{D}_i$). Let S be any natural deduction system. The equivalence class of $\mathfrak{D}_1$ under $\sim$ in S, denoted by $[\![\mathfrak{D}_1]\!]$, is defined as $[\![\mathfrak{D}_1]\!] = \{\mathfrak{D}_i \in S | \mathfrak{D}_1 \sim \mathfrak{D}_i\}$.

Then, the relation $\sim$ is clearly reflexive, symmetric, and transitive. Let $\hat{\mathfrak{D}}$ be an informal proof represented by a formal derivation $\mathfrak{D}$ in natural deduction. Prawitz's thesis and conjecture are summarized as below.

Prawitz's thesis: An admissible reduction procedure does not affect the identity of proofs represented by derivations in the same equivalence class.

The Conjecture for the Identity of Proofs(*CIP*): for any derivation $\mathfrak{D}_1$ and $\mathfrak{D}_2$, $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$ if and only if $\mathfrak{D}_1 \sim \mathfrak{D}_2$.

While Schroeder-Heister and Tranchini (2017) take Prawitz's thesis and *CIP*, they suggest the triviality test for evaluating whether a newly added reduction makes derivations representing different proofs belong to the same equivalence class. Let $\mathbb{R}$ be a set of reduction procedures. A set $\mathbb{R}'$ is an *extension* of $\mathbb{R}$ if $\mathbb{R}'$ results from $\mathbb{R}$ by adding reduction procedures which are closed under substitution in $\mathbb{R}'$. Let $\trianglerighteq$ be a reduction procedure not in a set $\mathbb{R}$ of reductions and $\mathbb{R}'$ be an extension of $\mathbb{R}$ by adding $\trianglerighteq$. We use an abbreviation '$S(\mathbb{R})$' for a natural deduction system S relative to $\mathbb{R}$. Let us consider the case that, for any three derivations Π_1, Π_2, and Π_3, Π_1 is reducible to Π_2 but not to Π_3 in $S(\mathbb{R})$ whereas Π_1 is reducible to both Π_2 and Π_3 in $S(\mathbb{R}')$. Suppose that $\hat{\Pi}_2 \neq \hat{\Pi}_3$. Then, $\Pi_2 \nsim \Pi_3$ by *CIP*. However, in $S(\mathbb{R}')$, $\Pi_1 \sim \Pi_2$ and $\Pi_1 \sim \Pi_3$, and so $\Pi_2 \sim \Pi_3$ in $S(\mathbb{R}')$.

While $\hat{\Pi}_2 \neq \hat{\Pi}_3$, it should be possible to show that Π_2 is not equivalent to Π_3. Since it is possible to show that $\Pi_2 \not\approx \Pi_3$ in $S(\mathbb{R})$ but not in $S(\mathbb{R}')$, we may say that an additional reduction $\trianglerighteq$ affects the identity of proofs. Following Prawitz's thesis, we may conclude that $\trianglerighteq$ is not an admissible reduction procedure. From this perspective, Schroeder-Heister and Tranchini (2017, p. 575) propose the triviality test for an admissible reduction process that a newly added reduction procedure should not trivialize the identity of proofs.

> A natural requirement of the addition of a new reduction could be that of not trivializing identity of proof, in the sense that it should always be possible to exhibit two derivations of the same conclusion belonging to two distinct equivalence classes.

We say that, for any derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ where $\hat{\mathfrak{D}}_1 \neq \hat{\mathfrak{D}}_2$ and so $\mathfrak{D}_1 \not\approx \mathfrak{D}_2$ in $S(\mathbb{R})$, a reduction process $\trianglerighteq$ *trivializes the identity of proofs* in S iff it is not possible to show in $S(\mathbb{R}')$ that $\mathfrak{D}_1 \not\approx \mathfrak{D}_2$; otherwise, $\trianglerighteq$ does *not trivialize the identity of proofs* in S.[3] Then, their triviality test can be summarized as below:

Triviality Test: Let S be any natural deduction system. Let $\trianglerighteq$ be a new reduction procedure not in $\mathbb{R}$ and $\mathbb{R}'$ be an extension of $\mathbb{R}$ by adding $\trianglerighteq$. $\trianglerighteq$ is an *admissible reduction* procedure for S iff $\trianglerighteq$ does not trivialize the identity of proofs in S.

[3] Another interpretation of the notion trivializing the identity of proofs may be that a newly added reduction trivializes the identity of proofs iff it is possible to show in $S(\mathbb{R}')$ that every provable derivation of the same conclusion is equivalent. This interpretation is stronger than the one we have proposed, making the triviality test more difficult to apply. Since Schroeder-Heister and Tranchini (2017, p. 575) explicitly state the meaning of 'not trivializing identity of proofs,' we will use the proposed stipulation in this paper.

Schroeder-Heister and Tranchini (2017, pp. 575-577) evaluate whether the Ekman reduction $\trianglerighteq_E$ is admissible in accordance with the triviality test. Let S_T be a natural deduction system containing $\wedge-$ and $\to-$rules. S_T has a set $\mathbb{R}_T$ of standard reductions for $\wedge$ and $\to$. $\mathbb{R}'_T$ is an extension of $\mathbb{R}_T$ by adding the Ekman reduction $\trianglerighteq_E$. If the Ekman reduction $\trianglerighteq_E$ does not trivialize the identity of proofs, $\trianglerighteq_E$ is an admissible reduction. They provide an example showing that it is not. First, they consider two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of the same formula which represent different proofs, i.e. $\hat{\mathfrak{D}}_1 \neq \hat{\mathfrak{D}}_2$, and so are not equivalent, i.e. $\mathfrak{D}_1 \not\sim \mathfrak{D}_2$. Then, they argue that whence $\trianglerighteq_E$ is added in $\mathbb{R}'_T$, it is not possible to show that $\mathfrak{D}_1 \not\sim \mathfrak{D}_2$ in $S_T(\mathbb{R}'_T)$.

For any two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of φ in S_T, Schroeder-Heister and Tranchini give a derivation Π_1 of $\varphi \wedge \varphi$ as below:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi]^1}{\varphi \wedge \varphi}\wedge I}{\varphi \to (\varphi \wedge \varphi)}\to I_{,1} \qquad \dfrac{\dfrac{\dfrac{[\varphi \wedge \varphi]^2}{\varphi}\wedge E_1}{(\varphi \wedge \varphi) \to \varphi}\to I_{,2} \quad \dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi & \varphi\end{matrix}}{\varphi \wedge \varphi}\wedge I}{\varphi}\to E}{\varphi \wedge \varphi}\to E}{\varphi}\wedge E_2$$

$S_T(\mathbb{R}_T)$ has a reduced derivation Π_2 on the left below from Π_1 by applying $\triangleright_{\to}$ and $\triangleright_{\wedge}$ with regard to $\wedge E_1-$rule. On the other hand, $S_T(\mathbb{R}'_T)$ has not only Π_2 but also Π_3 on the right below through the application of $\trianglerighteq_E$.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_1\\ \varphi & \varphi\end{matrix}}{\varphi \wedge \varphi}\wedge I}{\varphi}\wedge E_2 \qquad\qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi & \varphi\end{matrix}}{\varphi \wedge \varphi}\wedge I}{\varphi}\wedge E_2$$

A proof represented by derivations in $[\![\Pi_1]\!]$ should represent the same proof. Then, by the application of $\rhd_\wedge$ with regard to $\wedge E_2$−rule, we have Π_2' from Π_2 on the left below and Π_3' from Π_3 on the right below in $S_T(\mathbb{R}_T')$ as follows:

$$\begin{array}{cc} \mathfrak{D}_1 & \mathfrak{D}_2 \\ \varphi & \varphi \end{array}$$

By Definition 2.3, $\Pi_1 \sim \Pi_2'$ and $\Pi_1 \sim \Pi_3'$. Hence, $\Pi_2' \sim \Pi_3'$. This means that, for any derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of φ, $\mathfrak{D}_1 \sim \mathfrak{D}_2$, and so $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$. At this point, Schroeder-Heister and Tranchini's argument depends on an assumption that not every proof of φ is the same. That is, it is not the case that, for any derivation $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of φ in S_T, $\mathfrak{D}_1 \sim \mathfrak{D}_2$. If the assumption is true, there are derivations of φ in S_T that are not equivalent. However, since $\mathfrak{D}_1 \sim \mathfrak{D}_2$ in $S_T(\mathbb{R}_T')$, it is not possible to show in $S_T(\mathbb{R}_T')$ that $\mathfrak{D}_1 \not\sim \mathfrak{D}_2$. Therefore, the Ekman reduction $\trianglerighteq_E$ trivializes the identity of proofs. The triviality test says that $\trianglerighteq_E$ is not admissible.

As we have noted, the triviality test presumes Prawitz's thesis and the conjecture for the identity of proofs. If these are required for a correct natural deduction system, the triviality test would be an admissible reduction checker. However, there are still obstacles for the triviality test to be a promising criterion for the admissibility of reduction procedures.

3 Problems of the Triviality Test

There are three problems of the triviality test to be solved. A minor problem is that the triviality test is based on *CIP*. Prawitz (1971, p. 257) and Kreisel (1971, pp. 114-116) noticed that the right-to-left direction of *CIP* that, for any derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$, $\mathfrak{D}_1 \sim \mathfrak{D}_2$ implies $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$, is convincing, but the opposite direction that $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$

implies $\mathfrak{D}_1 \sim \mathfrak{D}_2$ is not. Provided that we have intuitively admissible reduction procedures, Prawitz's thesis says that an admissible reduction does not affect the identity of proofs represented by derivations in the same equivalence class. Equivalent derivations represent the same proof. That is, $\mathfrak{D}_1 \sim \mathfrak{D}_2$ implies $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$. On the other hand, the same proof can be represented by different derivations. It is not obvious that inequivalent derivations always represent different proofs. If the left-to-right direction of *CIP* is necessary for the triviality test, it should be explicated why $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$ implies $\mathfrak{D}_1 \sim \mathfrak{D}_2$ to have the triviality test as a promising admissible reduction checker.

It is unclear whether the triviality test must presume both directions of *CIP*. Schroeder-Heister and Tranchini seem to accept both directions. First, when Schroeder-Heister and Tranchini (2017, p. 574) introduce the perspective of the identity of proofs, they say, '[f]or simplicity, proofs might ... be considered to be equivalence classes with respect to derivations.' Although they intend to simplify the discussion for the triviality test, it can be interpreted that they admit both directions of *CIP*.

Moreover, when they introduce a *typical* example of two derivations which represent two distinct proofs, they use two derivations of the same conclusion which their assumptions are discharged at different places.

$$
\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \to \varphi}\,{\to}_{I,1}}{\varphi \to (\varphi \to \varphi)}\,{\to} I_{,\emptyset} \qquad \begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}}{\varphi \to \varphi}\,{\to} E
\qquad\qquad
\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \to \varphi}\,{\to} I_{,\emptyset}}{\varphi \to (\varphi \to \varphi)}\,{\to} I_{,1} \qquad \begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}}{\varphi \to \varphi}\,{\to} E
$$

Two derivations above belong to two different equivalent classes in the sense of Prawitz's equivalence relation. In the case of the empty discharge, the reduction process for $\rightarrow$ has the following form.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \psi\end{array}}{\varphi \rightarrow \psi}\rightarrow I_{,\emptyset} \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}}{\psi}\rightarrow E \qquad \rhd_{\rightarrow(\emptyset)} \qquad \begin{array}{c}\mathfrak{D}_1\\ \psi\end{array}$$

$\rhd_{\rightarrow(\emptyset)}$−reduction is an instance of $\rhd_{\rightarrow}$. Then, the reduced derivations below are obtained by $\rhd_{\rightarrow}$ respectively.

$$\dfrac{[\varphi]^1}{\varphi \rightarrow \varphi}\rightarrow I_{,1} \qquad \dfrac{\begin{array}{c}\mathfrak{D}\\ \varphi\end{array}}{\varphi \rightarrow \varphi}\rightarrow I_{,\emptyset}$$

The reduced derivations are different and so the original derivations are not equivalent.

If Schroeder-Heister and Tranchini interpret that the typical example shows that inequivalent derivations represent different proofs, their interpretation is based on the one half of Prawitz's *CIP* that $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$ implies $\mathfrak{D}_1 \sim \mathfrak{D}_2$, i.e. $\mathfrak{D}_1 \nsim \mathfrak{D}_2$ implies $\hat{\mathfrak{D}}_1 \neq \hat{\mathfrak{D}}_2$. Therefore, it is better to think that they accept the whole conjecture.

Prawitz (1971, pp. 256-257) introduced *CIP* as a philosophical consequence of the strong normalization theorem that every derivation is reducible to a unique normal derivation regardless of the order in which reductions are applied. If the strong normalization theorem holds in an intended system, every equivalent derivation has the same normal derivation. Inequivalent normal derivation cannot represent the same proof. Therefore, when the strong normalization theorem holds, inequivalent derivations represent distinct proofs, i.e. $\mathfrak{D}_1 \nsim \mathfrak{D}_2$ implies $\hat{\mathfrak{D}}_1 \neq \hat{\mathfrak{D}}_2$.

Unfortunately, there is no guarantee that the strong normalization theorem will be established if new rules are added in S or new reduction procedures are introduced. Also, we are in the context of finding the general criterion of an admissible reduction concerned with not only normalizable derivations but also paradoxical non-normalizable derivations. It is better not to accept the left-to-right direction of *CIP* that $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$ implies $\mathfrak{D}_1 \sim \mathfrak{D}_2$ unless there is a clear verification of the left-to-right direction.

We may partly accept *CIP*, i.e. $\mathfrak{D}_1 \sim \mathfrak{D}_2$ implies $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$, and use the triviality test. However, the triviality test has two more problems that have to be solved. The first problem is the obscurity of a test method. The triviality test uses modal notions such as 'possibility.' It is hard to say that the triviality test has a clear characterization of its application. The triviality test only states that when a new reduction is added, it *should be possible* to show that two derivations originally inequivalent still belong to different equivalent classes. Schroeder-Heister and Tranchini's argument finds a case that every derivation of the same formula becomes equivalent, but it does not the only way to establish the impossibility of showing that derivations representing different proofs belong to distinct equivalence classes. We may merely show that inequivalent derivations become equivalent after a new reduction process is added.

Moreover, it is not desirable to use modal notions as well as an informal notion, such as 'proof,' in order to give a clear characterization of the criterion for admissible reductions. The obscurity of the test method leads to an undesirable consequence that some standard reductions for primary constants are inadmissible because the triviality test does not explicate which derivations are fundamentally inequivalent.

Let S_t be a natural deduction system having only $\wedge-$rules. S_t has a set $\mathbb{R}_t$ of a reduction procedure which only has the following reduction process.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}\\ \varphi\wedge\varphi\end{array}}{\varphi}\wedge E_1 \quad \dfrac{\begin{array}{c}\mathfrak{D}\\ \varphi\wedge\varphi\end{array}}{\varphi}\wedge E_1}{\varphi\wedge\varphi}\wedge I \qquad \trianglerighteq_e \qquad \begin{array}{c}\mathfrak{D}\\ \varphi\wedge\varphi\end{array}$$

Suppose that, for any two inequivalent derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ ending with φ in $S_t(\mathbb{R}_t)$, there is a derivation Σ_1 as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}}{\varphi\wedge\varphi}\wedge I}{\varphi}\wedge E_1 \quad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}}{\varphi\wedge\varphi}\wedge I}{\varphi}\wedge E_1}{\varphi\wedge\varphi}\wedge I}{\varphi}\wedge E_2$$

In $S_t(\mathbb{R}_t)$, Σ_1 reduces to the derivation Σ_2 by $\triangleright_e$ on the left below. Let us consider the derivation Σ_3 on the right below.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}}{\varphi\wedge\varphi}\wedge I}{\varphi}\wedge E_2 \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array} \quad \begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array}}{\varphi\wedge\varphi}\wedge I}{\varphi}\wedge E_2$$

$\Sigma_1 \sim \Sigma_2$ but both Σ_1 and Σ_2 are not equivalent to Σ_3 in $S_t(\mathbb{R}_t)$ because Σ_1 and Σ_2 are not reducible to Σ_3 in $S_t(\mathbb{R}_t)$. Now, we add the standard reduction for $\wedge$, $\triangleright_\wedge$, to $\mathbb{R}_t$ and have $\mathbb{R}'_t$. Then, Σ_1 reduces to Σ_3 and Σ_3 reduces to the left derivation Σ_4 below. Also, Σ_2 reduces to the right derivation Σ_5 below.

$$\begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array} \qquad\qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}$$

Since $\Sigma_1 \sim \Sigma_4$ and $\Sigma_1 \sim \Sigma_5$ in $S_t(\mathbb{R}'_t)$, $\Sigma_4 \sim \Sigma_5$. However, $\mathfrak{D}_1 \nsim \mathfrak{D}_2$ in

$S_t(\mathbb{R}_t)$ and so $\Sigma_4 \nsim \Sigma_5$ in $S_t(\mathbb{R}_t)$. It is not possible to show in $S_t(\mathbb{R}'_t)$ that $\Sigma_4 \nsim \Sigma_5$, $\rhd_\wedge$ trivializes the identity of proofs in S_t and so is not an admissible reduction for S_t. Hence, the triviality test makes the standard reduction process $\rhd_\wedge$ inadmissible. Should we say that $\rhd_\wedge$ is an inadmissible reduction process?

The above result shows that the notion of an inequivalent derivation is relative to our choice of the set of reductions. If we use the triviality test with the set of reductions which already have inadmissible reductions, it can evaluate that even standard reduction procedures commonly considered admissible are inadmissible. In this regard, the base set of admissible reductions should be properly chosen. Unfortunately, at this point, the triviality test faces the second trouble.

The second problem is the circularity problem. The triviality test is (partially) based on *CIP*, especially that $\mathfrak{D}_1 \sim \mathfrak{D}_2$ implies $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$. The notion of an inequivalence relation between two derivations has the significant role of the triviality test. The matter is that the equivalence relation is dependent on which set of reductions we choose. The evaluation of whether a newly added reduction process trivializes the identity of proofs or not is relative to our choice of the base set of admissible reductions. By the way, since the triviality test should be an admissible reduction checker, the admissible reductions in the base set should also be evaluated by it. The circularity problem arises. If admissible reductions are dependent on our choice of reductions, we can simply choose without worrying about which reduction process is admissible. In the next section, we shall attempt to solve these problems and revise the triviality test.

4 The Spoiler Test and Solutions

In order to solve the circularity problem of the triviality test, we need to fix the base set of admissible reduction procedures. Although the Ekman reduction process gets rid of unnecessary steps, its role is different from the role of reductions for $\wedge$ and $\rightarrow$. Schroeder-Heister and Tranchini did not distinguish such roles. Unlike the Ekman reduction, the reduction procedures for $\wedge$ and $\rightarrow$ has the main role to eliminate a maximum formula in accordance with the inversion principle which is the core principle of Gentzen-Prawitz's natural deduction. In Section 4.1, we will distinguish between the standard and auxiliary reductions and define a *base set* of reductions. Then, the circularity problem will be solved. For the problem of the obscurity of a test method, we revise the triviality test and propose the spoiler test in Section 4.2.

4.1 The solution to the circularity problem: the standard and auxiliary reduction procedures.

When Gentzen (1935) introduces a natural deduction system, he explains the roles of introduction and elimination rules as below:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol.' (Gentzen, 1935, p. 80)

Gentzen's idea is often interpreted as the meaning of a logical constant is exhaustively determined by its introduction rule and determines its elimination rule.

Prawitz (1965) borrowed the idea and has developed it. To realize Gentzen's idea that an elimination rule is determined by the meaning of the conclusion of an introduction rule, there must be a certain requirement that fixes the elimination rule as the inverse of the corresponding introduction rule. For such requirement, Prawitz (1965, p. 33) suggests the inversion principle which states that whatever follows from a formula must follow from the direct ground for deriving that formula. We borrow his principle in the following way.

The Inversion Principle: Let $\mathfrak{D}_i$ be any immediate subderivation of an introduction rule for deriving the major premise of an elimination rule, $\mathfrak{D}_j$ be any derivation of minor premises of the elimination rule, and φ be any conclusion of the elimination rule. $\mathfrak{D}_i$ together with $\mathfrak{D}_j$ already derives φ without the application of the elimination rule. (i.e. any consequences of the major premise is derivable by $\mathfrak{D}_i$ together with $\mathfrak{D}_j$.)

The inversion principle reflects Gentzen's idea and says that nothing is gained by the application of an elimination rule when its major premise has been derived by means of an introduction rule. In order to show that a pair of introduction and elimination rules of each constant (or operator) satisfy the inversion principle, Prawitz (1965, pp. 35–38) proposes reduction procedures for logical constants. When a reduction procedure satisfies the inversion principle, it provides a method to eliminate a maximum formula. Thus, any reduction procedure eliminating a maximum formula in accordance with the inversion principle will be called a *standard reduction* procedure. The examples of standard reduction procedures are the reduction procedures for $\wedge$ and $\rightarrow$ introduced in Section 2.1.

The Ekman reduction is not a reduction which fits the inversion principle. Ekman maximum formula is not a maximum formula in the standard sense because it is neither a conclusion of an introduction rule nor a major premise of an elimination rule. Therefore, the Ekman reduction is not a standard reduction. Rather, it is a process to reduce the length of a derivation. We will call it an auxiliary reduction process.[4]

Since the inversion principle reflects Gentzen's idea, it explains not only the primary roles of introduction and elimination rules but also those of the standard reductions in natural deduction. With respect to the primary roles, Dummett (1991, p. 250) treats 'the eliminability of [maximum formulas] as a criterion for intrinsic harmony.' The intrinsic harmony is often regarded as a minimum requirement for acceptable rules. It can be defined via the inversion principle.

Definition 4.1. (Intrinsic Harmony) Let $\circ$ be a constant. Introduction and elimination rules for $\circ$ are *intrinsically harmonious* iff every pair consisting of an $\circ I-$ and $\circ E-$rules satisfies the inversion principle.

[4]The *degree* $d(\varphi)$ of a formula φ is defined by $d(\bot) = 0$, $d(\alpha) = 0$ for an atomic formula α, $d(\varphi \circ \psi) = d(\varphi) + d(\psi) + 1$ for binary operators $\circ$, $d(\circ\varphi) = d(\varphi) + 1$ for unary operators $\circ$. The *length* of a derivation $\mathfrak{D}$ is the number of formula occurrences. There are at least three sorts of auxiliary reductions which (i) lessens the degree of a major premise or (ii) lessens the length of a derivation, or (iii) changes the order of subderivations of a derivation. The example of (i) is a reduction process for the rule of *classical reductio* suggested by Prawitz (1965, p. 40). The Ekman reduction is the example of (ii). The example of (iii) is a permutation conversion introduced in Prawitz (1965, p. 51; 1971, p. 253). The target of these reductions (i), (ii), and (iii) are not to eliminate a maximum formula. Therefore, the standard reduction and auxiliary reductions are to be distinguished.

From the history of the development of natural deduction, it is convincing that we have minimal criteria of acceptable rules and of admissible reduction procedures through the inversion principle. Therefore, we say that, for any system S, the *base set* $\mathbb{R}$ of admissible reduction procedures of S consists of all standard reduction procedures for constants of intrinsically harmonious rules in S. Then, the circularity problem suggested in Section 3 is solved. When S_t has intrinsically harmonious rules for $\wedge$, it has the base set $\mathbb{R}_b$ consisting of the standard reduction for $\wedge$. Since $\Sigma_1 \sim \Sigma_3$ in $S_t(\mathbb{R}_b)$, we do not arrive at the conclusion that $\rhd_\wedge$ is inadmissible. Moreover, while an intended system has intrinsically harmonious rules, its base set of admissible reductions exists and the circularity problem does not occur.

4.2 The spoiler test

The circularity problem can be solved, but the triviality test still has an unclear characterization of its test method. For instance, as we have noted in footnote 3 of Section 2.2 and Section 3, it remains obscure whether we should show that every provable derivation of the same conclusion is equivalent in order to evaluate the triviality of a given reduction. Therefore, we propose the spoiler test as an improvement of the triviality test on Prawitz's perspective of the identity of proofs.

Although Prawitz (1971, pp. 256-257) suggests his thesis and *CIP* as the philosophical consequence of the strong normalization theorem, our criterion for admissible reductions should be applicable to a system that the strong normalization theorem does not hold. For instance, the derivation of a set-theoretic paradox introduced in Prawitz (1965, pp. 94-95) yields a non-terminating reduction sequence and so is not normalizable. The strong normalization theorem cannot be established in the system containing Prawitz's derivation of a set-theoretic paradox. However, it is hard to say that the derivation uses an inadmissible reduction process. The strong normalization theorem does not need to be requested for finding the criterion for admissible reduction procedures.

For persuing the admissible reduction checker based on the identity of proofs, especially the one half of *CIP*, i.e. $\mathfrak{D}_1 \sim \mathfrak{D}_2$ implies $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$, the Church-Rosser's confluence property (*CRCP*) is more suitable than the strong normalization.

The Church-Rosser's Confluence Property(*CRCP*): Let $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$ be any derivations in an intended natural deduction system. If $\mathfrak{D}_1 \succ \mathfrak{D}_2$ and $\mathfrak{D}_1 \succ \mathfrak{D}_3$ where $\mathfrak{D}_2 \nsim \mathfrak{D}_3$, then there is $\mathfrak{D}_4$ such that $\mathfrak{D}_2 \succ \mathfrak{D}_4$ and $\mathfrak{D}_3 \succ \mathfrak{D}_4$.

Church-Rosser theorem states something concerning reduction sequences in the lambda calculus, however the property was proved in an intuitionistic natural deduction system by Jean-Yves Girard (1987, pp. 135-150). Every normalizable derivation satisfying *CRCP* has at most one normal derivation. If two distinct derivations reduce to different normal derivations, they are not reduced from the same derivation. Moreover, a system having a paradoxical derivation which generates a non-terminating reduction sequence and so is not normalizable can satisfy *CRCP*. For instance, the derivation of a set-theoretic paradox introduced in Prawitz (1965, pp. 94-95) is not normalizable but it satisfies *CRCP* because *CRCP* does not require that every derivation should be normalizable.

While we only consider derivations satisfying *CRCP*, it is guaranteed that there are no distinct normal derivations reduced from the same derivation. Using *CRCP*, we introduce the spoiler test.

The Spoiler Test: Let S be any natural deduction system satisfying *CRCP*, and $\mathbb{R}$ be a base set of standard reductions for S. Let $\trianglerighteq$ be a reduction procedure not in $\mathbb{R}$ and $\mathbb{R}'$ be an extension of $\mathbb{R}$ by adding $\trianglerighteq$. $\trianglerighteq$ *spoils the identity of proofs in* $S(\mathbb{R})$ if there is a derivation in $S(\mathbb{R}')$ which does not satisfy *CRCP*; otherwise, it does not. $\trianglerighteq$ is an *admissible reduction* procedure in $S(\mathbb{R})$ if it does not spoiler the identity of proofs in $S(\mathbb{R})$; otherwise, it is inadmissible.

Any reduction procedures which spoil the identity of proofs are called a 'spoiler.' Let us consider the same system $S_T(\mathbb{R}_T)$ and $S_T(\mathbb{R}'_T)$ introduced in Section 2.2. $\mathbb{R}_T$ is the base system for S_T. $\mathbb{R}'_T$ is an extension of $\mathbb{R}_T$ by adding the Ekman reduction $\trianglerighteq_E$ to $\mathbb{R}_T$. We only need the right-to-left direction of *CIP* that $\mathfrak{D}_1 \sim \mathfrak{D}_2$ implies $\hat{\mathfrak{D}}_1 = \hat{\mathfrak{D}}_2$. Now, we have an argument that the Ekman reduction $\trianglerighteq_E$ is a spoiler and is inadmissible.

Let $\mathfrak{D}_1$ be any closed normal derivation of ψ. For any distinct closed normal derivations $\mathfrak{D}_2$ and $\mathfrak{D}_3$ of φ, we have the following derivation Δ_1 of φ.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_3\\ \varphi\end{matrix}}{\psi\to\varphi}\to I_{,\emptyset} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \psi\end{matrix}}{\varphi\to\psi}\to I_{,\emptyset} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi}\to E}{\varphi}\to E$$

The application of the Ekman reduction $\trianglerighteq_E$ to Δ_1 yields the left derivation Δ_2 below and the right derivation Δ_3 below is given by the application of $\triangleright_{\to(\emptyset)}$ to Δ_1.

$$\begin{array}{cc} \mathfrak{D}_2 & \mathfrak{D}_3 \\ \varphi & \varphi \end{array}$$

Since $\mathfrak{D}_2$ and $\mathfrak{D}_3$ are distinct normal derivations, there is no single derivation reducible from both $\mathfrak{D}_2$ and $\mathfrak{D}_3$. Therefore, there is a derivation in $S(\mathbb{R}')$ which does not satisfy *CRCP*. Therefore, the Ekman reduction process $\trianglerighteq_E$ is inadmissible because it is a spoiler.

5 An Objection: Crabbé's Case

Schroeder-Heister and Tranchini (2018) have considered an example observed by Marcel Crabbé which arises the problem of overgeneration of the proof-theoretic criterion for paradoxicality. So to speak, the case of Crabbé shows that the criterion for paradoxicality makes a non-paradoxical derivation paradoxical. They said that the triviality test fails to block the reduction procedures used in Crabbé's case and so it is a defect that the test cannot solve the overgeneration problem. If it is a defect of the triviality test, the spoiler test may have the same defect.

After we will introduce a brief summary of the development of the proof-theoretic criterion for paradoxicality in Section 5.1, we will see in Section 5.2 that every reduction used in Crabbé's case passes the spoiler test. Yet, we shall argue that not every problem of overgeneration is occurred by inadmissible reduction. When we focus on the method of evaluating admissible reduction, the spoiler test can be the criterion for admissible reduction procedures.

5.1 A brief summary of the development of the proof-theoretic criterion for paradoxicality

Prawitz (1965, Appendix B) first investigated that a derivation of $\perp$ from the set-theoretic paradox falls into a non-terminating reduction sequence, i.e. such derivation is not normalizable. Tennant (1982, p. 283) has claimed that the non-terminating reduction sequence is the distinguishing feature of the genuine paradoxes and proposed a proof-theoretic criterion for paradoxicality. Let M be any model and $\theta(M)$ be a set of sentences relative to M. He said,

> A set of sentences is paradoxical relative to M iff there is some proof of [$\perp$] from $\theta(M)$, involving those sentences in *id est* inferences, that has a looping reduction sequence.

His id est inferences may be any inferences having a formula interdeducible with its own negation (or its predication), such as inferences from φ to $\neg\varphi$ and $\neg\varphi$ to φ. Although his first stipulation of the criterion is not purely described in proof-theoretic fashion because of using the notion of a 'model,' Tennant's early version of the proof-theoretic criterion for paradoxicality, *PCP*, can be summarized as below.

Tennant's Proof-Theoretic Criterion for Paradoxicality(PCP_T): Let S be a natural deduction system relative to a set $\mathbb{R}$ of reduction procedures. Let $\mathfrak{D}$ be any derivation in S. $\mathfrak{D}$ is a *P-paradox* if and only if

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\perp$ [5],

[5] $\perp$ is not the only unacceptable conclusion. We can use a propositional vari-

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) applications of the reduction procedures in $\mathbb{R}$ to $\mathfrak{D}$ yield a non-terminating reduction sequence.[6]

PCP_T has a counterexample which raises the problem of over-generation. When PCP_T overgenerates, it includes a non-paradoxical derivation into the realm of P-paradoxes. Ekman's paradox suggested by Schroeder-Heister and Tranchini (2017) causes the overgeneration problem. Tennant (2017) has attempted to solve the problem and proposed an additional condition that every elimination rule is to be stated in generalized form. However, as Schroeder-Heister and Tranchini (2018) note, there is an Ekman-type paradox using generalized elimination rules which shows that Tennant's revised criterion overgenerates. We follow Schroeder-Heister and Tranchini's diagnosis that Ekman's paradox uses too loose reduction procedure. So instead of considering Tennant's additional criterion, we have the revised version of the proof-theoretic criterion for paradoxicality, PCP_R, by adding the following condition to PCP_T.

(iv) every reduction procedure in $\mathbb{R}$ is admissible.

Then, our discussions of the criterion for admissible reductions in the present paper can be melted into the discussions of PCP_R.

---

able p as an unacceptable conclusion while formulating Curry's paradox. For the examination of other cases, the reader can consult Tennant (1982)

[6] Tennant (1982, 1995) has introduced two types of a non-terminating reduction sequence; a looping reduction sequence and a spiraling reduction sequence. Tennant (1995, p. 207) conjectured that a looping reduction sequence is the main feature of the self-referential paradoxes whereas a spiraling reduction is generated by non-self-referential paradoxes, such as Yablo's paradox. For the problem of Tennant's conjecture, the reader can consult Choi (2021).

5.2 Should the spoiler test solve the overgeneration problem occurred by Crabbé's case?

The criterion for paradoxicality is one thing and the criterion for admissible reductions is another. Schroeder-Heister and Tranchini (2018) have regarded the triviality test as the solution for every overgeneration problem of PCP_R. The failure of solving the problem of overgeneration occurred by Crabbé's case becomes a defect of the triviality test.

On the other hand, we claim that the spoiler test need not be a general solution to all overgeneration problems. After we shall see that every reduction in Crabbé's case passes the spoiler test, we will argue that not every overgeneration problem is raised by inadmissible reductions.

To begin with, Crabbé's case may raise the overgeneration problem of PCP_R. It has rules for Zermelo's separation axiom. We propose the rules for Zermelo's separation axiom as the rules for $\in$ in the following way.

$$\dfrac{\overset{\mathfrak{D}_1}{t \in s} \quad \varphi[t/x]}{t \in \{x \in s | \varphi(x)\}} \in I \qquad \dfrac{t \in \{x \in s | \varphi(x)\}}{t \in s} \in E_1 \qquad \dfrac{t \in \{x \in s | \varphi(x)\}}{\varphi[t/x]} \in E_2$$

The standard reduction procedure $\rhd_{\in}$ for $\in I-$ and $\in E-$rules are as follows:

$$\dfrac{\dfrac{\overset{\mathfrak{D}_1}{t \in s} \quad \overset{\mathfrak{D}_2}{\varphi[t/x]}}{t \in \{x \in s | \varphi(x)\}} \in I}{t \in s} \in E_1 \quad \rhd_{\in_1} \quad \overset{\mathfrak{D}_1}{t \in s} \qquad \dfrac{\dfrac{\overset{\mathfrak{D}_1}{t \in s} \quad \overset{\mathfrak{D}_2}{\varphi[t/x]}}{t \in \{x \in s | \varphi(x)\}} \in I}{\varphi[t/x]} \in E_2 \quad \rhd_{\in_2} \quad \overset{\mathfrak{D}_2}{\varphi[t/x]}$$

We call both $\rhd_{\in_1}$ and $\rhd_{\in_2}$, 'Crabbé reduction.' For any set b, we define Z_b as a set $\{x \in b | \neg x \in x\}$. We take $\neg x \in x$ for φ in $\in I-$ and $\in E-$rules and for terms t and s we take Z_b and b respectively. Then, the following rules are the instances of $\in I-$ and $\in E-$rules.

$$\dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2 \\ Z_b \in b & \neg Z_b \in Z_b\end{array}}{Z_b \in \{x \in b | \neg x \in x\}} \in I \qquad \dfrac{Z_b \in \{x \in b | \neg x \in x\}}{Z_b \in b} \in E_1 \qquad \dfrac{Z_b \in \{x \in b | \neg x \in x\}}{\neg Z_b \in Z_b} \in E_2$$

Moreover, the instances of the reduction procedures $\rhd_{\in}$ can be suggested in a similar way. Let $S_{\in}$ be a system having $\in -$, $\rightarrow -$, and $\exists-$rules. Since $\rhd_{\rightarrow}$ and $\rhd_{\in}$ are standard reductions, $S_{\in}$ has a base set of reductions including $\rhd_{\rightarrow}$, $\rhd_{\exists}$, and $\rhd_{\in}$. Then, the following result is called, 'Crabbé's case.'

Lemma 5.1. *There is an open derivation Θ of $\bot$ from the assumption $[Z_b \in b]$ in $S_{\in}$ relative to $\mathbb{R}_{\in}$ which generates a non-terminating reduction sequence, and so is not normalizable.*

Proof. See Schroeder-Heister and Tranchini (2018) □

Proposition 5.2. *There is a closed derivation of $\neg\exists y(Z_y \in y)$ in $S_{\in}$.*

Proof. By using Lemma 5.1, it is easily proved. □

$\mathbb{R}_{\in}$ does not have auxiliary reductions but have standard reductions for $\rhd_{\rightarrow}$ and $\rhd_{\in}$. According to the spoiler test, all reductions in $\mathbb{R}_{\in}$ are admissible. Also, it is readily seen that every derivation in $S_{\in}$ satisfies *CRCP*.[7] Proposition 5.2 states that no set contains its own subset and is a common-sense result in a consistent Zermelo's paradox, but the derivation Θ of Lemma 5.1 satisfies PCP_R.

[7] For the proof, the reader can consult Girard (1987, pp. 135-150). The cases concerning $\rhd_{\in}$ are similar to the cases concerning $\rhd_{\wedge}$.

As Crabbé's case does not formulate a paradox, PCP_R makes it a P-paradox. Hence, Crabbé's case shows that PCP_R overgenerates and the spoiler test can not solve the overgeneration problem.

Since Schroeder-Heister and Tranchini (2018) think that the triviality test has a defect that it is unable to solve the overgeneration problem by Crabbé's case, they may say that the spoiler test has the same defect. However, the criterion for admissible reductions does not need to solve every problem of overgeneration by ruling out inadmissible reduction procedures. Not every overgeneration problem is caused by inadmissible reductions. For arbitrary φ, we consider the following odd rules for φ and their standard reduction $\rhd_{\varphi}$.

$$\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\varphi}\,\varphi I_{,1} \qquad\qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi \quad \varphi\end{matrix}}{\bot}\,\varphi E$$

The standard reduction for φ is stated as below.

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\varphi}\,\varphi I_{,1} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\bot}\,\varphi E \qquad \rhd_{\varphi} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \bot\end{matrix}$$

Let us consider a system having $\rightarrow -$ and $\varphi-$rules with their standard reductions. In the system, the following derivation is proved.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi]^1}{\bot}\,\varphi E}{\varphi \to \bot}\to I_{,1}}{\neg\varphi}\,def \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^2 \quad [\varphi]^2}{\bot}\,\varphi E}{\varphi \to \bot}\to I_{,2}}{\neg\varphi}\,def \qquad [\varphi]^3}{\bot}\to E}{\varphi}\,\varphi I_{,3}}{\bot}\to E$$

The derivation does not represent any paradox but satisfies PCP_T and PCP_R. It shows that both criteria overgenerate. However, we should not say that $\rhd_\varphi$ is the culprit of generating a non-terminating reduction because $\rhd_\varphi$ relies on the form of $\varphi I-$rule. $\varphi I-$rule already contains a constant that it is intended to introduce in its conclusion. A reduced derivation by $\rhd_\varphi$ still has a maximum formula φ. The reduction process does not terminate.

Similarly, the form of Crabbé reduction, $\rhd_\in$, relies on the form of $\in I-$rule. Since $\in I-$rule has the premise $\neg Z_b \in Z_b$ whose degree is greater than that of the conclusion $Z_b \in \{x \in b | \neg x \in x\}$, the reduced derivation has the formula $\neg Z_b \in Z_b$ by eliminating the maximum formula $Z_b \in \{x \in b | \neg x \in x\}$. Therefore, the real problem may not be Crabbé reduction itself but be the form of $\in I-$rule. From these cases, we conclude that not every problem of overgeneration is raised by inadmissible reductions. When we focus on the way to assess admissible reductions, the spoiler test can be regarded as a promising criterion for admissible reductions.[8]

[8]An anonymous reviewer pointed out that the purpose and conceptual ground of the spoiler test are different from those of the triviality test, and those differences are to be discussed. Unfortunately, because of the space limitation, those issues will be dealt with in the subsequent research.

6 Conclusion and Other Alternatives

In this article, we introduce Schroeder-Heister and Tranchini's triviality test and two main problems of the test: the obscurity of a test method and the circularity problem. In order to solve the two problems, we first have distinguished between the standard and auxiliary reduction procedures and proposed the way to have a base set of standard reduction procedures. Second, we have introduced the spoiler test as an improvement on the triviality test. The spoiler test can be a promising criterion for admissible reduction procedures.

Of course, there are other possibilities. It is often claimed that a natural deduction system is isomorphic to a sequent calculus system if the generalized eliminations are used. (Cf. Tennant (2002), Negri and von Plato (2001) and VonPlato (2011)). Especially, Negri and von Plato (2001, Ch.1 and Ch. 8) propose an inductive definition of the translation algorithm between an intuitionistic natural deduction system with generalized elimination rules and an intuitionistic sequent calculus system with independent contexts. Let Γ, Δ, Θ be a finite multiset, i.e. a list with multiplicity but no order, of assumptions in sequent calculus. We use a binary derivation symbol $\Rightarrow$ and '$\Gamma \Rightarrow \varphi$' means that the antecedent Γ derives the succedent φ. For convenience, we introduce the following left and right rules for $\to$, and cut-rule in sequent calculus.

$$\frac{\Gamma \Rightarrow \varphi \quad \psi, \Delta \Rightarrow \sigma}{\varphi \to \psi, \Gamma, \Delta \Rightarrow \sigma} L\to \qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi} R\to \qquad \frac{\Gamma \Rightarrow \varphi \quad \varphi, \Delta \Rightarrow \Psi}{\Gamma, \Delta \Rightarrow \Psi} Cut$$

φ in *Cut*−rule is called a *cut-formula*. The reduction process for $\to$ in the style of generalized elimination rule is as follows.

$$
\dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi\to\psi}\to I_{,1}\qquad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}\qquad \begin{matrix}[\psi]^1\\ \mathfrak{D}_3\\ \sigma\end{matrix}}{\sigma}\to E
\qquad \rhd_{\to} \qquad
\begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\\ \mathfrak{D}_3\\ \sigma\end{matrix}
$$

$\rhd_{\to}-$reduction in natural deduction is translated as below by Negri and von Plato's algorithm.

$$
\dfrac{\dfrac{\begin{matrix}\mathfrak{D}'_1\\ \varphi,\Gamma\Rightarrow\psi\end{matrix}}{\Gamma\Rightarrow\varphi\to\psi}R\to \quad \dfrac{\begin{matrix}\mathfrak{D}'_2\\ \Delta_1\Rightarrow\varphi\end{matrix}\quad \begin{matrix}\mathfrak{D}'_3\\ \psi,\Delta_2\Rightarrow\sigma\end{matrix}}{\varphi\to\psi,\Delta_1,\Delta_2\Rightarrow\sigma}L\to}{\Gamma,\Delta_1,\Delta_2\Rightarrow\sigma}Cut
\quad \text{reduces to} \quad
\dfrac{\dfrac{\begin{matrix}\mathfrak{D}'_2\\ \Delta_1\Rightarrow\varphi\end{matrix}\quad \begin{matrix}\mathfrak{D}'_1\\ \varphi,\Gamma\Rightarrow\psi\end{matrix}}{\Gamma,\Delta_1\Rightarrow\psi}Cut \quad \begin{matrix}\mathfrak{D}'_3\\ \psi,\Delta_2\Rightarrow\sigma\end{matrix}}{\Gamma,\Delta_1,\Delta_2\Rightarrow\sigma}Cut
$$

As the maximum formula $\varphi\to\psi$ is eliminated by $\rhd_{\to}-$reduction in the original derivation, the cut-formula in the translated derivation is removed by the above reduction. If a natural deduction with generalized elimination rules has an isomorphic translation to a sequent calculus, the isomorphic translation of normal derivations preserves the order of rules such that an introduction rule turns into corresponding right rule and an elimination rule into a left rule. The isomorphic translation guarantees the correspondence between normal and cut-free derivations. In this sense, if one of the main roles of an admissible reduction is to eliminate the maximum formula or to lessen the degree of it, an elimination of the cut-formula or a lessening of the degree of it plays the same role.

One of the interesting points is that Ekman's reduction stated in generalized form seems to be a cut-making process. Let us consider the following form of the Ekman reduction stated in generalized form with open assumptions.

$$\dfrac{[\psi \to \varphi] \quad \dfrac{[\varphi \to \psi] \quad \begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix} \quad [\psi]^1}{\psi}{\to E_{,1}} \quad \begin{matrix}[\varphi]^2\\ \mathfrak{D}_2\\ \sigma\end{matrix}}{\sigma}{\to E_{,2}} \qquad \trianglerighteq_{E_g} \qquad \begin{matrix}\mathfrak{D}_1\\ \varphi\\ \mathfrak{D}_2\\ \sigma\end{matrix}$$

The derivation is translated as below.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}'_1\\ \Gamma \Rightarrow \varphi\end{matrix} \quad \psi \Rightarrow \psi}{\Gamma, \varphi \to \psi \Rightarrow \psi}{L\to} \quad \begin{matrix}\mathfrak{D}'_2\\ \varphi, \Delta \Rightarrow \sigma\end{matrix}}{\varphi \to \psi, \psi \to \varphi, \Gamma, \Delta \Rightarrow \sigma}{L\to} \quad \textit{reduces to} \quad \dfrac{\begin{matrix}\mathfrak{D}'_1\\ \Gamma \Rightarrow \varphi\end{matrix} \quad \begin{matrix}\mathfrak{D}'_2\\ \varphi, \Delta \Rightarrow \sigma\end{matrix}}{\Gamma, \Delta \Rightarrow \sigma}{Cut}$$

Interestingly, the translated derivation on the left side has no cut-formula but after applying the reduction process, the cut-formula φ appears on the right side derivation. One who accepts that there is an isomorphic translation between natural deduction and sequent calculus can claim that the Ekman reduction is wrong because it is a maximum (or cut) formula-making process. On the other hand, one may say that the case shows the failure of the isomorphic translation between natural deduction and sequent calculus. Further investigation is needed to clarify the relation between two systems and to pursue the criterion for admissible reductions with regard to the isomorphic translation.

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증명의 동일성과 허용가능한 환원 절차의 기준

최 승 락

대그 프라위츠(1971)는 허용가능한 환원절차는 자연연역의 도출이 나타내는 증명들 간의 동일성에 아무런 영향을 미치지 않는다는 견해를 피력한 바 있다. 이러한 견해는 다음과 같은 그의 가설에 기반한다. 두 도출이 동일한 증명을 나타낼 때 오직 그 때만 이들은 동형이다. 여기서 동형이라는 것은 이들이 가까운 환원가능성 관계에 관해 재귀적이고 이행적이며 대칭적이라는 것이다.

슈레더하이스터와 트란키니(2017)는 프라위츠의 가설을 받아들이고 허용가능한 환원절차의 기준으로 사소성 테스트를 제안한다. 이 글에서, 필자는 사소성 테스트의 두 가지 문제점을 지적할 것이다. 첫 번째는, 허용가능한 환원절차를 평가하는 방식이 모호하다는 점이다. 그리고 두 번째 문제는 사소성 테스트가 이미 허용가능한 환원절차의 집합을 가정있다는 점이다. 이러한 문제점의 해결책으로, 필자는 스포일러 테스트를 제안할 것이다. 이는 사소성 테스트의 문제점에 영향을 받지 않을 뿐만 아니라 허용가능한 환원절차의 역할도 할 것이다. 마지막으로 필자는 크랩 사례에 의해 제기될 수 있는 스포일러 테스트의 가능한 문제점을 고려할 것이다.

주요어: 허용가능한 환원절차, 에크만 역설, 증명간의 동일성, 크랩 사례, 사소성 테스트.