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Involutive extensions of fixpointed mianorm-based logics*

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【Abstract】 Yang recently investigated standard completeness for fixpointed mianorm-based logics. This paper extends these logics to involutive ones. More exactly, first involutive fixpointed mianorm-based logics and their algebraic semantics are introduced. Next, some examples of involutive fixpointed mianorms are considered. Finally, standard completeness for the systems are provided.

【Key words】 involution; substructural logic; fuzzy logic; fixpoint; mianorm

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1. Introduction

One of important research areas in substructural (core) fuzzy logic, called (core) semilinear logic, is to consider involutive extensions of basic substructural fuzzy logic systems. For instance, after Yang (2016) introduced standard algebraic semantics[1)] for the fuzzy logic system MIAL (Mianorm logic), he (2017a) investigated such semantics for its involutive extension IMIAL (Involutive MIAL).[2)]

Some involutive extensions having standard algebraic semantics require the fixpoint axiom t ↔ f (FP) as well as the axioms for involution. For instance, the involutive extension of UML (Uinorm mingle logic) having such semantics, i.e., the system IUML (Involutive uninorm mingle logic), requires both fixpoint and involution axioms. This means that the system UML with the involution axiom (RM^T) is not standard complete, i.e, not complete with respect to (w.r.t.) standard algebraic semantics (see Yang (2019)).

In this respect, it is interesting to study involutive extensions of basic substructural fuzzy logics with fixpoint and their standard completeness. Here we note

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1) Standard algebraic semantics is algebraic semantics on the real unit interval [0, 1]. Fuzzy (semilinear resp) logics having with standard algebraic semantics are called *core* fuzzy (semilinear resp) logics and such completeness is called *standard* completeness.

2) For more such examples, see Yang (2020a).

that Yang (2020b) recently investigated fixpointed semilinear logics and their standard completeness. This gives a natural question as follows.

Q: Can we provide standard completeness for the involutive extension of those fixpointed semilinear logics?

The answer is 'yes'. To verify it, first we introduce involutive extensions of the fixpointed semilinear logics introduced in Yang (2020b) and consider their algebraic completeness. We next provide some examples of involutive fixpointed mianorms. Finally, we study standard completeness for those logics.

2. Logics and algebraic semantics

As preliminaries, we discuss involutive extensions of the fixpointed mianorm-based logics introduced in Yang (2020b) and their algebraic semantics. We base these logics on a countable propositional language, which has *Fm*, a set of formulas, being inductively built from *VAR*, a set of propositional variables, binary connectives $\vee$, $\wedge$, &, $\rightarrow$, $\rightsquigarrow$, and constants F, T, t, f, with defined connectives: $\neg\alpha := \alpha \rightarrow f$, $\sim\alpha := \alpha \rightsquigarrow f$, $\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$, $\alpha_t := \alpha \wedge t$, ${}^n\alpha := \alpha$ & $(\alpha$

& ⋯ & (α & α) ⋯)), n factors; and α^n := ((⋯ (α & α) & ⋯ & α) & α, n factors.

Definition 2.1 *(i)* (Yang 2016; 2017a) The following are axiom schemes and rules for MIAL.:

$(\alpha \wedge \beta) \to \alpha$, $(\alpha \wedge \beta) \to \beta$;
$((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma))$;
$\alpha \to (\alpha \vee \beta)$, $\beta \to (\alpha \vee \beta)$;
$((\alpha \to \gamma) \wedge (\beta \to \gamma)) \to ((\alpha \vee \beta) \to \gamma)$;
$\mathrm{F} \to \alpha$; $(\mathrm{t} \to \alpha) \leftrightarrow \alpha$; $(\alpha_{\mathrm{t}} \,\&\, \beta_{\mathrm{t}}) \to (\alpha \wedge \beta)$;
$\alpha \to (\beta \to (\beta \,\&\, \alpha))$; $\alpha \to (\beta \rightsquigarrow (\alpha \,\&\, \beta))$;
$(\beta \,\&\, (\alpha \,\&\, (\alpha \to (\beta \to \gamma)))) \to \gamma$;
$((\alpha \,\&\, (\alpha \rightsquigarrow (\beta \to \gamma))) \,\&\, \beta) \to \gamma$;
$((\alpha \to (\alpha \,\&\, (\alpha \to \beta))) \& (\beta \to \gamma)) \to (\alpha \to \gamma)$;
$((\alpha \rightsquigarrow ((\alpha \rightsquigarrow \beta) \,\&\, \alpha))) \,\&\, (\beta \to \gamma)) \to (\alpha \rightsquigarrow \gamma)$;
$(\alpha \to \beta)_{\mathrm{t}} \vee ((\delta \,\&\, \varepsilon) \to (\delta \,\&\, (\varepsilon \,\&\, (\beta \to \alpha)_{\mathrm{t}})))$ $(\mathrm{PLa}_{\delta,\varepsilon})$;
$(\alpha \to \beta)_{\mathrm{t}} \vee ((\delta \,\&\, \varepsilon) \to ((\delta \,\&\, (\beta \to \alpha)_{\mathrm{t}}) \,\&\, \varepsilon))$ $(\mathrm{PLa'}_{\delta,\varepsilon})$;
$(\alpha \to \beta)_{\mathrm{t}} \vee (\delta \to (\varepsilon \to ((\varepsilon \,\&\, \delta) \,\&\, (\beta \to \alpha)_{\mathrm{t}})))$ $(\mathrm{PLb}_{\delta,\varepsilon})$;
$(\alpha \to \beta)_{\mathrm{t}} \vee (\delta \to (\varepsilon \rightsquigarrow ((\varepsilon \,\&\, \delta) \,\&\, (\beta \to \alpha)_{\mathrm{t}})))$ $(\mathrm{PLb'}_{\delta,\varepsilon})$;
$\alpha \to \beta, \alpha \vdash \beta$; $\alpha \vdash \alpha_{\mathrm{t}}$;
$\alpha \vdash (\delta \,\&\, \varepsilon) \to (\delta \,\&\, (\varepsilon \,\&\, \alpha))$;
$\alpha \vdash (\delta \,\&\, \varepsilon) \to ((\delta \,\&\, \alpha) \,\&\, \varepsilon)$;
$\alpha \vdash \delta \to (\varepsilon \to ((\varepsilon \,\&\, \delta) \,\&\, \alpha))$;
$\alpha \vdash \delta \to (\varepsilon \rightsquigarrow ((\delta \,\&\, \varepsilon) \,\&\, \alpha))$.

IMIAL, the involutive MIAL, is MIAL plus
$\sim\neg\alpha \to \alpha$ (double negation elimination, DNE(1));
$\neg\sim\alpha \to \alpha$ (DNE(2)).

FIMIAL, the fixpointed IMIAL. is IMIAL plus t ↔ f (*F*).

(*ii*) Consider the following structural axioms:

(contraction, *c*) $\alpha \to (\alpha \ \& \ \alpha)$

(expansion, *p*) $(\alpha \ \& \ \alpha) \to \alpha$

(left *n*–contraction, c^l_n) ${}^{n-1}\alpha \leftrightarrow {}^{n}\alpha$, $2 \leq n$

(right *n*–contraction, c^r_n) $\alpha^{n-1} \to \alpha^{n}$, $2 \leq n$

(left *n*–mingle, m^l_n) ${}^{n}\alpha \leftrightarrow {}^{n-1}\alpha$, $2 \leq n$

(right *n*–mingle, m^r_n) $\alpha^{n} \to \alpha^{n-1}$, $2 \leq n$.

$FIMIAL_S$, $S \subseteq \{c, p, c^l_n, c^r_n, m^l_n, m^r_n\}$, is an involutive fuzzy logic extending FIMIAL.

We henceforth fix S as a subset such that $S \subseteq \{c, p, c^l_n, c^r_n, m^l_n, m^r_n\}$.

Definition 2.2 FILs = $\{FIMIAL_S: S \subseteq \{c, p, c^l_n, c^r_n, m^l_n, m^r_n\}\}$

Remark 2.3 By dropping the axiom *F* from FIMIAL, we have the system IMIAL introduced in Yang (2017a); by eliminating the axioms DNE(1) and DNE(2) from FIMIAL (IMIAL resp), we obtain the system FMIAL (MIAL resp), see Yang (2020b; 2016). Note that $FMIAL_S$ has been introduced in Yang (2020b).

Henceforth, we, for convenience, use the notations, "~," "¬," "→," "⇝," "∨," and "∧ " both as propositional connectives and as algebraic operators.

We next introduce algebraic structures characterizing $FIMIAL_S$ ∈ FILs. Let ¬x, ~x, and x_t be x → f, x ⇝ f, and x ∧ t, respectively.

Definition 2.4 (*i*) (FIMIAL-algebra, Yang (2017a)) A *FIMIAL-algebra* is an algebra **A** = (A, ⊥, ⊤, t, f, ∨, ∧, *, →, ⇝) such that:

(Ⅰ) (A, ⊤, ⊥, ∨, ∧) is a bounded lattice.

(Ⅱ) (A, *, t, f) is a fixpointed unital groupoid.

(Ⅲ) for all x, y, z ∈ A, x * y ≤ z iff y ≤ x → z iff x ≤ y ⇝ z (residuation).

(IV) for all x, y, z, w ∈ A,

- $t \leq (x \to y)_t \vee ((z*w) \to (z*(w*(y \to x)_t)))$ $(PLa_{\delta,\varepsilon}{}^{A})$
- $t \leq (x \to y)_t \vee ((z*w) \to ((z*(y \to x)_t)*w))$ $(PLa'_{\delta,\varepsilon}{}^{A})$
- $t \leq (x \to y)_t \vee (z \to (w \to ((w*z)*(y \to x)_t)))$ $(PLb_{\delta,\varepsilon}{}^{A})$
- $t \leq (x \to y)_t \vee (z \to (w \rightsquigarrow ((w*z)*(y \to x)_t)))$ $(PLb'_{\delta,\varepsilon}{}^{A})$.

(V) ~¬x = x $(DNE(1)^{A})$, ¬~x = x $(DNE(2)^{A})$.

(*ii*) ($FIMIAL_S$-algebras) The following are the inequations for the axioms introduced in Definition 2.1 (*ii*): for all x ∈ A,

- $x \leq x * x$ (c^{A}); • $x * x \leq x$ (p^{A})
- ${}^{n-1}x \leq {}^{n}x$, $2 \leq n$, $(c^{l}_{n}{}^{A})$; • $x^{n-1} \leq x^{n}$, $2 \leq n$, $(c^{r}_{n}{}^{A})$;
- ${}^{n}x \leq {}^{n-1}x$, $2 \leq n$, $(m^{l}_{n}{}^{A})$; • $x^{n} \leq x^{n-1}$, $2 \leq n$, $(m^{r}_{n}{}^{A})$.

$FIMIAL_S$-algebras, $S \subseteq \{c^{A}, p^{A}, c^{l}_{n}{}^{A}, c^{r}_{n}{}^{A}, m^{l}_{n}{}^{A}, m^{r}_{n}{}^{A}\}$, are defined along with corresponding inequations.

We call a $FIMIAL_S$-algebra *linearly ordered* if $x \leq y$ or $y \leq x$ for each pair x, y. We define an A-valuation as a map $v : Fm \rightarrow A$ such that $v(\#(\alpha_1, \cdots, \alpha_n)) = \#^A(v(\alpha_1), \cdots, v(\alpha_n))$, where $\# \in \{\rightarrow, \rightsquigarrow, \&, \wedge, \vee, T, F, t, f\}$ and $\#^A \in \{\rightarrow, \rightsquigarrow, *, \wedge, \vee, \top, \bot, t, f\}$. We say that a formula α is *valid* in A in case $t \leq v(\alpha)$ for all A-valuation v and that an A-valuation v is an A-*model* of T in case $t \leq v(\alpha)$ for all $\alpha \in T$.

Theorem 2.5 (Completeness) Let T be a theory over $FIMIAL_S \in FILs$ and α a formula. $T \vdash_{FIMIALS} \alpha$ iff for all linearly ordered $FIMIAL_S$-algebras A and an A-valuation v, if v is an A-model of T, then $t \leq v(\alpha)$.

Proof: This claim is a corollary of Theorem 3.1.8 in Cintula & Noguera (2011). □

3. Involutive fmianorms and their examples

In this section, by *0*, *1*, $\mathcal{E}$ and ϑ, we denote $\bot$, $\top$, identity t, and any f, respectively, on the real unit interval [0, 1]. We first note that standard $FIMIAL_S$-algebras are $FIMIAL_S$-algebras over the real unit interval [0, 1].

Definition 3.1 (*i*) (Mianorm, Yang (2016)) A *mianorm* is a map $\bigcirc$: $[0, 1]^2 \rightarrow [0, 1]$ such that for all x, y, z $\in$ [0, 1] and for some $\varepsilon \in [0, 1]$:

- $\varepsilon \bigcirc x = x \bigcirc \varepsilon = x$ (identity), and
- if $x \leq y$, then $x \bigcirc z \leq y \bigcirc z$ and $z \bigcirc x \leq z \bigcirc y$ (monotonicity).

(*ii*) (Fixpointed mianorm, Yang (2020b)) A *fixpointed mianorm* (*fmianorm* for short) is a mianorm with $\ni = \varepsilon$.

(*iii*) (Yang (2020b)) *S-fmianorms*, S $\subseteq \{c^A, p^A, c^{l\,A}_n, c^{r\,A}_n, m^{l\,A}_n, m^{r\,A}_n\}$, are defined, along with their corresponding inequations.

(*iv*) ($FIMIAL_S$-mianorm) Involutive S-fmianorms are S-fmianorms satisfying $(DNE(1)^A)$ and $(DNE(2)^A)$. We call these fmianorms *$FIMIAL_S$-mianorms*.

A mianorm $\bigcirc$ is called *conjunctive* if $0 \bigcirc 1 = 1 \bigcirc 0 = 0$.

Note that an involutive pair of negations ($\neg$, ~) is called *cyclic* if $\neg x = {\sim}x$ for all x in [0, 1]. A residuated pair of implications (\, /) is *involutive* if it further satisfies $(DNE(1)^A)$ and $(DNE(2)^A)$. Note also that the operator * of any standard IMIAL-algebra is a conjunctive mianorm with identity ε and involutively residuated pair of implications (\, /); conversely, any involutively residuated mianorm gives rise to an IMIAL-algebra.

For a cyclic involutive negation $\neg$ and fixpointed identity ε (= $\neg\varepsilon$)) $\in$ (0, 1), we introduce some examples of $FIMIAL_S$−mianorms.

Example 3.2 Let $\neg$ be a cyclic involutive negation and identity ε (= $\neg\varepsilon$)) $\in$ (0, 1).

(*i*) A conjunctive fmianorm $\bigcirc_1$ and its involutively residuated pair ($\backslash_1$, $/_1$) are given by:

$$x \bigcirc_1 y = \begin{cases} \max(0, x+y-\varepsilon) & \text{if } y \leq \neg x \text{ and } x, y \leq \varepsilon; \\ \min(x, y) & \text{if } y \leq \neg x \text{ and } x \leq y \text{ or } y = 0; \\ \varepsilon & \text{if } y \leq \neg x \text{ and otherwise}; \\ \min(1, x+y-\varepsilon) & \text{if } y > \neg x \text{ and } \varepsilon \leq x, y; \\ \max(x, y) & \text{otherwise}, \end{cases}$$

$$x \backslash_1 y = \neg(\neg y \bigcirc_1 x),\ y /_1 x = \neg(x \bigcirc_1 \neg y).$$

(*ii*) A conjunctive contractive fmianorm $\bigcirc_2$ and its involutively residuated pair ($\backslash_2$, $/_2$) are given by:

$$x \bigcirc_2 y = \begin{cases} \min(x,y) & \text{if } y \leq \neg x \text{ and } x \leq \varepsilon \text{ or } y = 0; \\ \varepsilon & \text{if } y \leq \neg x \text{ and otherwise}; \\ \min(1, x+y-\varepsilon) & \text{if } y > \neg x \text{ and } \varepsilon \leq x, y; \\ \max(x, y) & \text{otherwise}, \end{cases}$$

$$x \backslash_2 y = \neg(\neg y \bigcirc_2 x),\ y /_2 x = \neg(x \bigcirc_2 \neg y).$$

(*iii*) A conjunctive expansive fmianorm $\bigcirc_3$ and its involutively residuated pair ($\backslash_3$, $/_3$) are given by:

$$x \bigcirc_3 y = \begin{cases} \max(0, x+y-\varepsilon) & \text{if } y \le \neg x \text{ and } x, y \le \varepsilon; \\ \min(x,y) & \text{if } y \le \neg x \text{ and } x \le y \text{ or } y = 0; \\ \varepsilon & \text{if } y \le \neg x \text{ and otherwise;} \\ \max(x, y) & \text{otherwise,} \end{cases}$$

$$x \backslash_3 y = \neg(\neg y \bigcirc_3 x),\ y /_3 x = \neg(x \bigcirc_3 \neg y).$$

(*iv*) A conjunctive right (left) 3−contractive fmianorm $\bigcirc_4$ and its involutively residuated pair ($\backslash_4$, $/_4$) are given by:

$$x \bigcirc_4 y = \begin{cases} 0 & \text{if } y \le \neg x \text{ and } x, y \le \varepsilon; \\ \min(x, y) & \text{if } y \le \neg x \text{ and } x \le y \text{ or } y = 0; \\ \varepsilon & \text{if } y \le \neg x \text{ and otherwise;} \\ \max(x, y) & \text{otherwise,} \end{cases}$$

$$x \backslash_4 y = \neg(\neg y \bigcirc_4 x),\ y /_4 x = \neg(x \bigcirc_4 \neg y).$$

(*v*) A conjunctive right (left) 3−mingle fmianorm $\bigcirc_5$ and its involutively residuated pair ($\backslash_5$, $/_5$) are given by:

$$x \bigcirc_5 y = \begin{cases} \min(x, y) & \text{if } y \le \neg x \text{ and } x \le \varepsilon \text{ or } y = 0; \\ \varepsilon & \text{if } y \le \neg x \text{ and otherwise;} \\ 1 & \text{if } y > \neg x \text{ and } \varepsilon < x, y; \\ \max(x, y) & \text{otherwise,} \end{cases}$$

$$x \backslash_5 y = \neg(\neg y \bigcirc_5 x),\ y /_5 x = \neg(x \bigcirc_5 \neg y).$$

4. Standard completeness

Here we provide standard completeness for $\mathrm{FIMIAL_S}$ $\in$ FILs via a model-theoretic construction introduced in Yang (2017a; 2017b). For convenience, we add the 'less than or equal to' relation symbol "$\leq$" to such algebras.

Theorem 4.1 (Yang (2020b)) For every finite or countable linearly ordered $\mathrm{FMIAL_S}$-algebra A = (A, $\leq_A$, $\bot$, $\top$, f, t, $\vee$, $\wedge$, $*$, $\rightarrow$, $\rightsquigarrow$), we can build a countable ordered set U, a binary operation $\bigcirc$, and a map *h* from A into U such that the following conditions hold:

(Ⅰ) U is densely ordered, and has a minimum Min, a maximum Max, and special elements Ɛ, Э.

(Ⅱ) (U, $\bigcirc$, $\leq$, Ɛ, Э) is a linearly ordered, monotonic, unital fixpointed groupoid.

(Ⅲ) $\bigcirc$ is conjunctive and left-continuous.

(Ⅳ) *h* is an embedding function of the algebra (A, $\leq_A$, $\bot$, $\top$, f, t, $\vee$, $\wedge$, $*$) into (U, $\leq$, Min, Max, Э, Ɛ, max, min, $\bigcirc$), and for all k, l $\in$ A, h(k $\rightarrow$ l) and h(k $\rightsquigarrow$ l) are the residuated pair of h(k) and h(l) in (U, $\leq$, Min, Max, Э, Ɛ, max, min, $\bigcirc$).

(Ⅴ) $\bigcirc$ satisfies structural properties corresponding to those of $*$.

Lemma 4.2 For every finite or countable linearly ordered $\mathrm{FIMIAL_S}$-algebra A = (A, $\leq_A$, $\bot$, $\top$, f, t, $\vee$, $\wedge$, $*$, $\rightarrow$, $\rightsquigarrow$), we can build a countable ordered set U, a binary operation ◯, and a function *h* from A into U such that the conditions (I) to (V) in Theorem 4.1 and the following condition hold:

(VI) For all x ∈ U, x satisfies (DNE(1)^{A}) and (DNE(2)^{A}).

Proof: For convenience, let us suppose that A is a subset of Q ∩ [0, 1] having a finite or countable number of elements, where 1 and 0 are greatest and least elements, respectively. First notice that, for $FMIAL_S$, a monotonic linearly ordered unital fixpointed groupoid (U, ◯, ≤, Ɛ, Э) is defined as follows:

U = {(0, 0)} ∪ {(k,x): k ∈ A \ {0 (= ⊥)} and x ∈ Q ∩ (0,k]};

for (k, x), (l, y) ∈ U,

(k, x) ≤ (l, y) iff either k $<_A$ l, or k $=_A$ l and x ≤ y;

for FMIAL and $FMIAL_A$, A ∈ {c, c^l_n, c^r_n, where $2 \leq n$},

(k,x) $◯_{U1}$ (l,y) = max{(k,x),(l,y)} if k∗l = k∨l, k$\neq_A$l, and (k,x) ≤ Ɛ or (l,y) ≤ Ɛ;
min{(k,x),(l,y)} if k∗l = k∧z, and (k, x) ≤ Ɛ or (l, y) ≤ Ɛ;
(k∗l, k∗l) otherwise, and

for $FMIAL_B$, B $\in$ $\{p_n, m^l_2, m^r_2\}$,

$$(k,x) \bigcirc_{U2} (l,y) = \begin{cases} \max\{(k,x),(l,y)\} & \text{if } k*l = k\vee l, \text{ and } (k,x) > \varepsilon \text{ or } (l,y) > \varepsilon; \\ \min\{(k,x),(l,y)\} & \text{if } k*l = k\wedge l, \text{ and } (k,x) \leq \varepsilon \text{ or } (l,y) \leq \varepsilon; \\ (k*l,\ k*l) & \text{otherwise.} \end{cases}$$

For convenience, we henceforth delete the index A in $\leq_A$ and $=_A$, in case we do not have to distinguish them.

Next notice that, for $FIMIAL_B$, k^+ denotes the successor of k if it exists, otherwise $k^+ = k$, for each k $\in$ A. Define $\neg k := k \setminus \ni$ and $\sim k := \ni / k$. Then, the pair of negations $(\neg, \sim)$ in A is involutive. Hence, we have that: $k = (\neg l)^+$ iff $l = (\neg k)^+$ and $k = (\sim l)^+$ iff $l = (\sim k)^+$; moreover, if $k < k^+$, then $(\neg(k^+))^+ = \neg k$ and $(\sim(k^+))^+ = \sim k$. Here, we use V below in place of the U above. Let $(V, \leq)$ be the linearly ordered set, defined by

$$V = \{(k, k): k \in A\} \cup \{(k, x): \exists k' \in A \text{ such that } k = k'^+ > k', \text{ and } x \in Q \cap (0, k)\},$$

and $\leq$ being the corresponding lexicographic ordering as above. Then, it suffices to check the condition (VI).

Now, we define new operations $\circledast_{V1}$ on V, based on

$\bigcirc_{U1}$, and $\circledast_{V2}$ on V, based on $\bigcirc_{U2}$, as follows:[3)]

$$\begin{aligned}(k,x) \circledast (l,y) = \min\{\ni,(k,x)\bigcirc(l,y)\}\ &\text{if } k=(\neg l)^{+} \text{ and } a/b+a'/b' \leq 1, \text{ where } x = ka/b \text{ and } y=la'/b', \text{ or } k < (\neg l)^{+}; \text{ or}\\ &\text{if } k=({\sim}l)^{+} \text{ and } a/b+a'/b' \leq 1, \text{ where } x = ka/b \text{ and } y=la'/b', \text{ or } k < ({\sim}l)^{+}; \text{ or}\\ (k,x) \bigcirc (l,y)\ &\text{otherwise.}\end{aligned}$$

Note that $\circledast_{V1}$ is for FIMIAL and FIMIAL_{A}, A $\in$ $\{c, c^{l}_{n}, c^{r}_{n}$, where $2 \leq n\}$ and $\circledast_{V2}$ is for FIMIAL_{B}, B $\in$ $\{p, m^{l}_{2}, m^{r}_{2}\}$,

For conditions (I), (II), (III), (IV), (VI), and (F^{A}), see Proposition 2 in Yang (2017a) and Proposition 3.2 in Yang (2017b). Hence, we need to consider the condition (V). For the conditions for FIMIAL_{c} and FIMIAL_{p}, see Proposition 3.2 in Yang (2017b) and Theorem 4 in Yang (2017a). For the other conditions for $\mathrm{FIMIAL}_{c^{r}_{n}}$, $\mathrm{FIMIAL}_{c^{l}_{n}}$, $\mathrm{FIMIAL}_{m^{r}_{n}}$, and $\mathrm{FIMIAL}_{m^{l}_{n}}$, see Proposition 3 in Yang (2019).

Note that the other logics are obtained by combining some of the structural axioms c, p, c^{l}_{n}, c^{r}_{n}, m^{l}_{n}, and m^{r}_{n}. We can similarly prove additional properties. As an example, we consider $\mathrm{FIMIAL}_{c^{r}_{2}m^{r}_{3}}$, which has $\circledast_{V1}$. For $\mathrm{FIMIAL}_{c^{r}_{2}m^{r}_{3}}$, we need to further prove right 3-mingle

3) This definition was introduced in Yang (2017a). For convenience, we drop the indices V_1 and V_2 whenever we do not have to distinguish them.

and 2-contractive property for (k, x) $\in$ U. Note that it forms right n-potency, i.e., $(k, x)^n = (k, x)^{n-1}$, for $3 \le n$, and the right n-potency was proved in Yang (2020a). Thus, it suffices to show that $(k, x) \le (k, x)^2$ and $(k, x)^3 = (k, x)^2$.

Case 1. $k = (\sim k)^+$ and $2a/b \le 1$, where $x = ka/b$, or $k < (\sim k)^+$.

Subcase 1.1. $k = k^2$. Since $t < k$ is not the case, we have $k = k^2 \le t = f < (\sim k)^+$ and thus $(k, x) \circledast_{V1} (k, x) = \min\{\ni, (k, x) \bigcirc_{V1} (k, x)\} = (k, x) \bigcirc_{V1} (k, x) = (k, x)$; therefore, $(k, x) \le (k, x)^2$. Moreover, since $(k, x) \circledast_{V1} (k, x) = (k, x) \bigcirc_{V1} (k, x) = (k, x)^2 \bigcirc_{V1} (k, x) = (k, x)^2 \circledast_{V1} (k, x)$, we further have that $(k, x)^2 = (k, x)^3$.

Subcase 1.2. $k \neq k^2$. This cannot be the case since the condition implies that $k^2 < k < t$, which contradicts the supposition that k is right 2-contractive, i.e., $k \le k^2$.

Case 2. $k = (\neg k)^+$ and $2a/b \le 1$, where $x = ka/b$, or $k < (\neg k)^+$. The proof is similar to that of Case 1.

Case 3. Otherwise. The proof reduces to that of the right 3-mingle and 2-contractive property for $\mathrm{FMIAL}_{c^r 2m^r 3}$, which is proved in Proposition 4.2 in Yang (2020b). □

Lemma 4.3 Every countable linearly ordered $\mathrm{FIMIAL_S}$-algebra can be embedded into a standard

$FIMIAL_S$-algebra.

Proof: Let V, A, etc. be as in Lemma 4.2. Since (V, ≤) is a linearly-ordered countable, dense set with minimum and maximum, it is order isomorphic to ([0, 1] ∩ Q, ≤). We take g as such an isomorphism. If (I) to (VI) hold true, letting for a, b ∈ [0, 1], a ⊛′ b = $g(g^{-1}(a) \circledast g^{-1}(b))$, and, for all k ∈ A, h′(k) = g(h(k)), we have that [0, 1] ∩ Q, ≤, 0, 1, Ɛ, ∋, ⊛′, h′ satisfy the conditions (I) to (VI) whenever V, ≤, Max, Min, Ɛ, ∋, ⊛, and h do. This means that we can assume that V = [0, 1] ∩ Q and ≤ = ≤, without loss of generality.

For a, b ∈ [0, 1], let us define ⊛″ as follows,

$$a \circledast'' b = \sup_{x \in U: x \leq a} \sup_{y \in U: y \leq b} x \circledast y.$$

By this definition, we can easily show that it satisfies monotonicity, identity, fixpoint, and idempotence. Furthermore, it follows from the definition that ⊛″ is conjunctive, i.e., 0 ⊛″ 1 = 0. The left-continuity of ⊛″ can be proved as in Proposition 2 in Yang (2017a).

For the other structural properties of ⊛″, here as an example we prove the right 3-mingle and 2-contractive property as in Lemma 4.2. Suppose that in the unit interval $\langle x_i : i \in N \rangle$ is an increasing sequence of reals, where $\sup\{x_i : i \in N\} = x$. Then, we obtain that $x^{n-1} = \sup\{a^{n-1} : a \in [0, 1] \cap Q, a \leq x\}$ and $x^n = \sup\{a^n : a \in [0, 1] \cap Q, a \leq x\}$. For the right 3-mingle and 2-contractive property of $\circledast''$, we need to show that $x \leq x^2$ and $x^3 = x^2$. Since $a \leq a^2$, we have that $\sup\{a : a \in [0, 1] \cap Q, a \leq x\} \leq \sup\{a^2 : a \in [0, 1] \cap Q, a \leq x\}$. Thus, we get that $x \leq x^2$. Similarly, since $x^3 = x^2$, we obtain that $\sup\{a^3 : a \in [0, 1] \cap Q, a \leq x\} = \sup\{a^2 : a \in [0, 1] \cap Q, a \leq x\}$; therefore, $x^3 = x^2$.

As an easy consequence of the definition, we have that $\circledast''$ extends $\circledast$. By (I) to (VI) of Lemma 4.2, h is an embedding function of $(A, \leq_A, \top, \bot, t, f, \wedge, \vee, *)$ into $([0, 1], \leq, 1, 0, \mathcal{E}, \ni, \min, \max, \circledast'')$. Moreover, $\circledast''$ has a residuated pair of implications, calling it $(\rightharpoondown, \rightharpoonup)$.

Finally we prove that for $x, y \in A$, $h(x \setminus y) = h(x) \rightharpoondown h(y)$ and $h(y / x) = h(x) \rightharpoonup h(y)$. By (IV), $h(x \setminus y)$ and $h(y / x)$ are the residuated pair of implications of $h(x)$ and $h(y)$ in $([0, 1] \cap Q, \leq, 1, 0, \mathcal{E}, \ni, \min, \max, \circledast)$. Thus

$$h(x) \circledast'' h(x \setminus y) = h(x) \circledast h(x \setminus y) \leq h(y), \text{ and}$$
$$h(y / x) \circledast'' h(x) = h(y / x) \circledast h(x) \leq h(y).$$

For the first case, suppose toward contradiction that there is $a > h(x \setminus y)$ such that $h(x) \circledast'' a \leq h(y)$. Since $[0, 1] \cap Q$ is dense in $[0, 1]$, there is $q \in [0,$

1] ∩ Q such that $h(x \setminus y) < q \leq a$. Hence $h(x) \circledast$ " $q = h(x) \circledast q \leq h(y)$, contradicting the condition (IV). The proof for the second is analogous. □

Theorem 4.4 (Strong standard completeness) For $\mathrm{FIMIAL_S}$, $S \subseteq \{c, p, c^l_n, c^r_n, m^l_n, m^r_n\}$, the following are equivalent:

(1) $T \vdash_{\mathrm{FIMIALS}} \alpha$.

(2) For every standard $\mathrm{FIMIAL_S}$-algebra and evaluation v, if $v(\beta) \geq \varepsilon$ for all $\beta \in T$, then $v(\alpha) \geq \varepsilon$.

Proof: The (1)-to-(2) direction is easy. For the (2)-to-(1) direction, let α be a formula such that $T \nvdash_{\mathrm{FIMIALS}} \alpha$, A a linearly ordered $\mathrm{FIMIAL_S}$-algebra, and v an evaluation in A such that $v(\beta) \geq t$ for all $\beta \in T$ and $v(\alpha) < t$. Let h' be the embedding function of A into the standard $\mathrm{FIMIAL_S}$-algebra as in proof of Lemma 4.3. Then, $h' \circledast v$ is an evaluation into the standard $\mathrm{FIMIAL_S}$-algebra such that $h' \circledast v(\beta) \geq \varepsilon$ and yet $h' \circledast v(\alpha) < \varepsilon$. □

5. Concluding remark

We investigated involutive extensions of fixpoint mianorm-based logics. After introducing their some examples, we provided standard completeness results for unknown such logics via Yang's construction. Note, however, that this construction does not work for involutive extensions of mianorm-based logics, i.e., non-fixpointed involutive extensions (see Yang (2017b)). To introduce such semantics for non-fixpointed involutive extensions remains a problem.

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고정점을 갖는 미아놈 논리의 누승적 확장

양 은 석

최근 고정점을 갖는 미아놈 논리가 연구되었다. 이 논문은 그러한 논리를 누승적인 논리로 확장한다. 이를 위하여 먼저 고정점을 갖는 누승적 미아놈 논리와 그러한 논리의 대수적 의미론을 소개한다. 다음으로 고정점을 갖는 누승적 미아놈의 몇몇 예를 소개한다. 마지막으로 누승적 논리 체계들이 표준적으로 완전하다는 것 즉 단위 실수 [0, 1]에서 완전하다는 것을 보인다.

주요어: 누승, 준구조 논리, 퍼지 논리, 고정점, 미아놈.