논리연구 23-1(2020) pp. 1-24

Mianorm-based Logics with right and left n-potency axioms*

Eunsuk Yang

【Abstract】 This paper deals with mianorm-based logics with right and left n-potency axioms and their fixpointed involutive extensions. For this, first, right and left n-potent logic systems based on *mianorms*, their corresponding algebraic structures, and their algebraic completeness results are discussed. Next, completeness with respect to algebras whose lattice reduct is [0, 1], known as s*tandard completeness,* is established for these systems via Yang's construction in the style of Jenei–Montagna. Finally, further standard completeness results are introduced for their fixpointed involutive extensions.

【Key Words】 fuzzy logic, substructural logic, mianorm, algebraic completeness, standard completeness, n-potency.

Received: Nov. 05, 2019. Revised: Feb. 05, 2020. Accepted: Feb. 20, 2020.

* This research was supported by "Research Base Construction Fund Support Program" funded by Jeonbuk National University in 2019. I must thank the referees for their helpful comments.

1. Introduction

This paper is a contribution of a standard completeness for basic substructural fuzzy logics with *n*-potency axioms. For this, let us first recall some historical facts associated with such logics. After Esteva and Godo (2001) introduced Monoidal t-norm logic **MTL** as the most basic t-norm[1)] logic, fuzzy logic systems based on more general structures have been introduced. For instance, uninorms-based logics were introduced by Metcalfe and Montagna (2007).[2)] Micanorm-based and mianorm-based logics have been introduced by Yang (2015; 2016).[3)]

In particular, standard completeness for basic fuzzy logics and their corresponding *n*-potent logics have been established: Jenei and Montagna (2002) provided standard completeness for **MTL** and then Ciabattoni, Esteva, and Godo (2002) introduced such completeness for **CnMTL** (**MTL** with *n*-contraction axiom (Cn) $\phi^{n-1} \rightarrow \phi^n$, for $2 < n$)[4)]; similarly, standard completeness for Uninorm logic **UL** and **PnUL** (**UL** with n-potency axiom (Pn) were established by Metcalfe and Montagna (2007) and Wang (2012), respectively;[5)] standard completeness for Micanorm logic

---

1) T-norms are commutative, associative, increasing, binary functions with identity 1 on the real unit interval [0, 1].

2) Uninorms are a generalization of t-norms where the identity can lie anywhere in [0, 1].

3) Micanorms are uninorms dropping associativity and mianorms are micanorms eliminating commutativity.

4) This system is equivalent to **PnMTL** (**MTL** with *n*-potency axiom (Pn) $\phi^{n-1} \leftrightarrow \phi^n$, for $2 < n$) because the right-to-left direction of (Pn) is provable in **CnMTL**.

MICAL (= SL^1_e) and **PnMICAL** (**MICAL** with *n*-potency axiom) were established by Cintula et al. (2013) and Yang (2015), respectively; and standard completeness for Mianorm logic **MIAL** (= SL^1) and **PnMIAL** (**MIAL** with *n*-potency axiom) were established by Cintula et al. (2013) and Yang (2016), respectivel y.[6)]

Recently, Yang (2019) realized that the *n*-potency axiom can be divided into the *n*-contraction axiom (Cn) above and *n*-mingle axiom (Pn) $\phi^n \rightarrow \phi^{n-1}$, for $2 < n$, and these two axioms can be further divided into left and right ones in the context of non-commutative logic.[7)] (Let $\phi^n := ((\cdots(\phi \,\&\, \phi) \,\&\, \cdots \phi) \,\&\, \phi$, n factors, and ${}^n\phi := \phi \,\&\, (\phi \,\&\, \cdots\& \,(\phi \,\&\, \phi) \cdots))$, n factors. Then we can distinguished the right and left axioms.) According to this distinction, the system **PnMIAL** introduced in Yang (2016) has to be divided right and left ones.[8)] Then, a natural question arises as follows:

---

5) The latter system was first denoted by **CnUL**. But in order to eliminate any unnecessary confusion, here we denote it by **PnUL** and Similarly for **CnMICAL** and **CnMIAL**.

6) The systems **MICAL** and **MIAL** were first introduced as SL^1_e and SL^1, respectively, in Cintual et al. (2013) and Horĉík (2011). (Note that, as a referee commented, the latter names do not show that those logics are core fuzzy logics, whereas the former names show it.) To emphasize that the systems are based on micanorms and mianorms, respectively, here we describe them as the micanorm-based logic **MICAL** and the mianorm-based logic **MIAL**. For some more detailed reasons, see Yang (2016).

7) As mentioned in Yang (2019), Hori et al. (1994) first introduced *n*-contraction and n-mingle axioms and Baldi (2014) introduced Wang's **CnUL**, for n > 2, as **UL** with both the n-contraction and n-mingle axioms.

8) Note that **PnMIAL** introduced in Yang (2016) corresponds to the **MIAL** with right *n*-potency axiom below.

Can we introduce mianorm-based logics with left and right *n*-potency axioms?

It seems that the answer is yes because the n-potency axiom can be divided into left and right *n*-contractive and *n*-mingle axioms and these systems have been investigated in Yang (2019). However, unfortunately this idea was not verified in it. Here we verify this fact. More precisely, we introduced the **MIAL** with left and right *n*-potency axioms and provide standard completeness for them.

The paper is organized as follows. In Section 2, we discuss the mianorm-based logic **MIAL** with right and left *n*-potency axioms along with their corresponding algebras. In Section 3, we establish standard completeness for those logics using the Jenei-Montagna-style construction introduced in Ciabattoni, Esteva, & Godo (2002) and Jenei & Montagna (2002). Especially, standard completeness is established for these systems via Yang's construction introduced in Yang (2015; 2016). In Section 4, further standard completeness results are introduced for their *fixpointed* involutive extensions.[9)]

For convenience, we shall adopt notations and terminology similar to those in Cintula (2006), Hájek (1998), Metcalfe & Montagna (2007), Yang (2015; 2016, 2017a; 2017b), and assume reader familiarity with them (together with the results found

---

9) Note that Yang already considered such extensions in Yang (2017b). However, in standard completeness, he did not provide an exact proof for left and right n-potencies. Thus, here we reconsider those systems.

therein).

2. Syntax

We base some axiomatic extensions of the mianorm logic **MIAL** on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, $\Rightarrow$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with a defined connective:[10)]

df1. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We moreover define $\phi_{\mathbf{t}} := \phi \wedge \mathbf{t}$. For the rest of this paper, we use the customary notations and terminology, and the axiom systems to provide a consequence relation.

We start with the following axiomatization of **MIAL**, the most basic fuzzy logic introduced here.

Definition 2.1 (Yang (2016)) **MIAL** consists of the following axiom schemes and rules:

A1. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ ($\wedge$-elimination, $\wedge$-E)

A2. $((\phi\rightarrow\psi)\wedge(\phi\rightarrow\chi)) \rightarrow (\phi\rightarrow(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)

A3. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A4. $((\phi\rightarrow\chi)\wedge(\psi\rightarrow\chi)) \rightarrow ((\phi\vee\psi)\rightarrow\chi)$ ($\vee$-elimination, $\vee$-E)

---

10) The notation $\Rightarrow$ (as one of implications) is in general expressed by squigarrow $\twoheadrightarrow$ or leftarrow $\leftarrow$ and the notation & (as strong conjunction) is often expressed by fusion $\circ$.

A5. $\mathbf{F} \to \phi$ (ex falso quadlibet, EF)

A6. $(\mathbf{t} \to \phi) \leftrightarrow \phi$ (push and pop, PP)

A7. $\phi \to (\psi \to (\psi \& \phi))$ (&-adjunction$_{\to}$, &-Adj$_{\to}$)

A8. $\phi \to (\psi \Rightarrow (\phi \& \psi))$ (&-Adj$_{\Rightarrow}$)

A9. $(\phi_{\mathbf{t}} \& \psi_{\mathbf{t}}) \to (\phi \wedge \psi)$ (&∧)

A10. $(\psi \& (\phi \& (\phi \to (\psi \to \chi)))) \to \chi$ (residuation, Res')

A11. $((\phi \& (\phi \Rightarrow (\psi \to \chi))) \& \psi) \to \chi$ (Res'$_{\Rightarrow}$)

A12. $((\phi \to (\phi \& (\phi \to \psi))) \& (\psi \to \chi)) \to (\phi \to \chi)$ (T')

A13. $((\phi \Rightarrow ((\phi \Rightarrow \psi) \& \phi))) \& (\psi \to \chi)) \to (\phi \Rightarrow \chi)$ (T'$_{\Rightarrow}$)

A14. $(\phi \to \psi)_{\mathbf{t}} \vee ((\delta \& \varepsilon) \to (\delta \& (\varepsilon \& (\psi \to \phi)_{\mathbf{t}})))$ (PL$\alpha_{\delta,\varepsilon}$)

A15. $(\phi \to \psi)_{\mathbf{t}} \vee ((\delta \& \varepsilon) \to ((\delta \& (\psi \to \phi)_{\mathbf{t}}) \& \varepsilon))$ (PL$\alpha'_{\delta,\varepsilon}$)

A16. $(\phi \to \psi)_{\mathbf{t}} \vee (\delta \to (\varepsilon \to ((\varepsilon \& \delta) \& (\psi \to \phi)_{\mathbf{t}})))$ (PL$\beta_{\delta,\varepsilon}$)

A17. $(\phi \to \psi)_{\mathbf{t}} \vee (\delta \to (\varepsilon \Rightarrow ((\varepsilon \& \delta) \& (\psi \to \phi)_{\mathbf{t}})))$ (PL$\beta'_{\delta,\varepsilon}$)

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi \vdash \phi_{\mathbf{t}}$ (adj$_u$)

$\phi \vdash (\delta \& \varepsilon) \to (\delta \& (\varepsilon \& \phi))$ (α)

$\phi \vdash (\delta \& \varepsilon) \to ((\delta \& \phi) \& \varepsilon)$ (α')

$\phi \vdash \delta \to (\varepsilon \to ((\varepsilon \& \delta) \& \phi))$ (β)

$\phi \vdash \delta \to (\varepsilon \Rightarrow ((\delta \& \varepsilon) \& \phi))$ (β')

Right and left *n*-potent fuzzy logics are defined by extending **MIAL** with suitable axiom schemes. Especially, we introduce the following extensions.

Definition 2.2 A logic is an axiomatic extension (extension for short) of an arbitrary logic **L** if and only if (iff) it results from **L** by adding axiom schemes. Especially, we introduce two particular extensions of **MIAL**.

- Right n-potent mianorm logic
  $\mathbf{P^r_n MIAL}$ is **MIAL** plus (P^r_n) $\phi^n \leftrightarrow \phi^{n-1}$, $2 \leq n$.
- Left n-potent mianorm logic
  $\mathbf{P^l_n MIAL}$ is **MIAL** plus (P^l_n) ${}^n\phi \leftrightarrow {}^{n-1}\phi$, $2 \leq n$.

For easy reference, we let Ls be the set of the fuzzy logics defined in Definition 2.2.

Definition 2.3 Ls = $\{\mathbf{P^r_n MIAL}, \mathbf{P^l_n MIAL}\}$

A *theory* over L $\in$ Ls is a set T of formulas. A *proof* in a theory over L is a sequence of formulas each of whose members is either an axiom of L or a member of T or follows from some preceding members of the sequence using a rule of L. T $\vdash \phi$, more exactly T $\vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. A theory T is *inconsistent* if T $\vdash$ **F**; otherwise it is *consistent*.

The deduction theorem for L is as follows:

Proposition 2.4 (Cintula et al. (2013; 2015)) Let T be a theory, and ϕ, ψ formulas. T $\cup$ $\{\phi\}$ $\vdash_L \psi$ iff T $\vdash_L \gamma(\phi) \rightarrow \psi$ for some $\gamma \in \Pi(bDT^*)$.[11]

For convenience, "$\sim$," "$\wedge$," "$\vee$," "$\rightarrow$," and "$\Rightarrow$" are used

11) For γ and $\Pi(bDT^*)$, see Cintula et al. (2013; 2015) and Yang (2015).

ambiguously as propositional connectives and as algebraic operators, but context should clarify their meaning.

Suitable algebraic structures for L $\in$ Ls are obtained as a subvariety of the variety of residuated lattice-ordered groupoids (briefly rlu-groupoids) in the sense of Galatos et al. (2007).

Definition 2.5 (Yang (2016)) (i) A *pointed bounded rlu-groupoid* is a structure $\mathbf{A} = (A, \top, \bot, t, f, \wedge, \vee, *, \rightarrow, \Rightarrow)$ such that:

(Ⅰ) $(A, \top, \bot, \wedge, \vee)$ is a bounded lattice with top element $\top$ and bottom element $\bot$.

(Ⅱ) $(A, *, t)$ is a groupoid with unit.

(Ⅲ) $y \leq x \rightarrow z$ iff $x * y \leq z$ iff $x \leq y \Rightarrow z$, for all x, y, z $\in$ A (residuation).

Since the class of pointed, bounded rlu-groupoids characterizes the system SL, we henceforth call these groupoids *SL-algebras*.

(ii) Let x_t be $x \wedge t$. An *MIAL-algebra* is an SL-algebra satisfying: for all x, y, z, w $\in$ A,

- $t \leq (x \rightarrow y)_t \vee ((z * w) \rightarrow (z * (w*(y\rightarrow x)_t)))$ $(PL\alpha_{\delta,\varepsilon}^{\mathbf{A}})$
- $t \leq (x \rightarrow y)_t \vee ((z * w) \rightarrow ((z*(y\rightarrow x)_t) * w))$ $(PL\alpha'_{\delta,\varepsilon}{}^{\mathbf{A}})$
- $t \leq (x \rightarrow y)_t \vee (z \rightarrow (w \rightarrow ((w*z) * (y\rightarrow x)_t)))$ $(PL\beta_{\delta,\varepsilon}^{\mathbf{A}})$
- $t \leq (x \rightarrow y)_t \vee (z \rightarrow (w \Rightarrow ((w*z)*(y\rightarrow x)_t)))$ $(PL\beta'_{\delta,\varepsilon}{}^{\mathbf{A}})$.

L-algebras the class of which characterizes L are defined as follows.

Definition 2.6 (L-algebras) The algebraic (in)equations corresponding to the structural axioms introduced in Definition 2.2 are defined as follows: for all x $\in$ A,

- $x^n = x^{n-1}$, $2 \leq n$, $(P^{r}_{n}{}^{\mathbf{A}})$
- ${}^{n-1}x = {}^{n}x$, $2 \leq n$, $(P^{l}_{n}{}^{\mathbf{A}})$

A *P^r_nMIAL-algebra* is an MIAL-algebra satisfying $(P^{r}_{n}{}^{\mathbf{A}})$ and a *P^l_nMIAL-algebra* is an MIAL-algebra satisfying $(P^{l}_{n}{}^{\mathbf{A}})$. We call these algebras *L-algebras*.

An L-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., x $\leq$ y or y $\leq$ x (equivalently, x $\wedge$ y = x or x $\wedge$ y = y) for each pair x, y.

Definition 2.7 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function v : *Fm* $\rightarrow$ $\mathcal{A}$ satisfying: $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$, $v(\phi \Rightarrow \psi) = v(\phi) \Rightarrow v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{F}) = \bot$, $v(\mathbf{f}) = f$, (and hence $v(\mathbf{T}) = \top$ and $v(\mathbf{t}) = t$).

Definition 2.8 Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and K a class of L-algebras.

(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq t$ for each $\phi \in$ T. We denote the class of $\mathcal{A}$-models of T, by *Mod(T, $\mathcal{A}$)*.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by T $\vDash_K$ ϕ, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ {ϕ}, $\mathcal{A}$) for each $\mathcal{A}$ $\in$ K.

Definition 2.9 (**L**-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is an ***L**-algebra* iff, whenever ϕ is L-provable in T (i.e. T $\vdash_L$ ϕ, L an L logic), it is a semantic consequence of T w.r.t. the set {$\mathcal{A}$} (i.e. T$\vDash_{\{A\}}$ ϕ). By *MOD*$^{(l)}$*(L)*, we denote the class of (linearly ordered) **L**-algebras. Finally, we write T $\vDash^{(l)}_L$ ϕ in place of T $\vDash_{MOD^{(l)}(L)}$ ϕ.

Theorem 2.10 (Strong completeness) Let T be a theory, and ϕ a formula. T $\vdash_L$ ϕ iff T $\vDash_L$ ϕ iff T $\vDash^1_L$ ϕ.

Proof: We obtain this theorem as a corollary of Theorem 3.1.8 in Cintula & Noguera (2011). □

3. Standard completeness

Here we establish standard completeness for L $\in$ Ls by use of the Jenei-Montagna-style construction introduced in Yang (2015; 2016). First note the following facts.

Fact 3.1 (Yang (2016)) For every finite or countable linearly ordered **MIAL**-algebra **A** = (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *, $\to$, $\Rightarrow$), there is a countable ordered set X, a binary operation $\circ$, and a map *h* from A into X such that the following conditions hold:

(Ⅰ) X is densely ordered, and has a maximum Max, a minimum Min, and special elements e, ∂.

(Ⅱ) (X, $\circ$, $\leq$, e) is a linearly ordered, monotonic, groupoid with unit.

(Ⅲ) $\circ$ is conjunctive and left-continuous w.r.t. the order topology on (X, $\leq$).

(Ⅳ) h is an embedding of the structure (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *) into (X, $\leq$, Max, Min, e, ∂, min, max, $\circ$), and for all m, n $\in$ A, h(m $\rightarrow$ n) and h(m $\Rightarrow$ n) are the residuated pair of h(m) and h(n) in (X, $\leq$, Max, Min, e, ∂, max, min, $\circ$).

Fact 3.2 (Strong standard completeness, Yang (2016)) For **MIAL**, the following are equivalent:

(1) T $\vdash_{\mathbf{MIAL}}$ ϕ.

(2) For every standard **MIAL**-algebra and evaluation v, if $v(\psi) \geq e$ for all $\psi \in$ T, then $v(\phi) \geq e$.

As in Fact 3.1, we then show that finite or countable, linearly ordered L-algebras are embeddable into a densely ordered L-algebra.

Proposition 3.3 For every finite or countable linearly ordered L-algebra **A** = (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *, $\rightarrow$, $\Rightarrow$), there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the conditions (I) to (IV) in Fact 3.1 and the following condition hold:

(V) ○ satisfies right (left resp) *n*-potent property corresponding to *.

Proof: For convenience, we assume A as a subset of **Q** ∩ [0, 1] with a finite or countable number of elements, where 0 and 1 are least and greatest elements, respectively, each of which corresponds to ⊤ and ⊥, respectively. Let

$$X = \{(m, x): m \in A \setminus \{0\ (= \bot)\} \text{ and } x \in \mathbf{Q} \cap (0, m] \cup \{(0, 0)\}.$$

For (m, x), (n, y) ∈ X, we define:

$$(m, x) \leq (n, y) \text{ iff either } m <_A n, \text{ or } m =_A n \text{ and } x \leq y.$$

For convenience, we henceforth drop the index A in $\leq_A$ and $=_A$, if we need not distinguish them. Context should clarify the intention.

Now we need to define ○ for $\mathbf{P^r_nMIAL}$ and $\mathbf{P^l_nMIAL}$. For $\mathbf{P^r_nMIAL}$ and $\mathbf{P^l_nMIAL}$, 3 ≤ *n*, the definition of ○ is as follows. For (m, x), (n, y) ∈ X,

$$(m,x) \circ (n,y) = \begin{cases} \max\{(m,x), (n,y)\} & \text{if } m*n = m \vee n,\ m \neq_A n, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \\ \min\{(m,x), (n,y)\} & \text{if } m * n = m \wedge z, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \\ (m * n, m * n) & \text{otherwise.} \end{cases}$$

For $\mathbf{P^r_2MIAL}$ (= $\mathbf{P^l_2MIAL}$), the definition of ○ is as follows. For (m, x), (n, y) ∈ X,

$$(m,x) \circ (n,y) = \begin{cases} \max\{(m,x), (n,y)\} & \text{if } m * n = m \vee n, \text{ and } (m, x) \succ e \text{ or } (n, y) \succ e; \\ \min\{(m,x), (n,y)\} & \text{if } m * n = m \wedge z, \text{ and } (m, x) \preceq e \text{ or } (n, y) \preceq e; \\ (m * n, m * n) & \text{otherwise.} \end{cases}$$

For the proof of the conditions (I) to (IV), see Proposition 2 in Yang (2016). We prove the condition (V).

For $\mathbf{P^l_nMIAL}$, we have to further prove left *n*-potency, i.e., ${}^{n}(m, x) = {}^{n-1}(m, x)$ for $2 \leq n$ and for (m, x) ∈ X.

Case 1. e ≺ (m, x). If ${}^{2}m = m$, then (m, x) ○ (m, x) = (m, x) and thus ${}^{n-1}(m, x) = {}^{n}(m, x)$. Otherwise, we have $t < m < {}^{2}m$ and thus $(m, x) \circ (m, x) = ({}^{2}m, {}^{2}m)$. Hence, similarly we further have that $(m, x) \circ ({}^{2}m, {}^{2}m) = ({}^{3}m, {}^{3}m)$ and thus ${}^{n-1}(m, x) = ({}^{n-1}m, {}^{n-1}m)$ and ${}^{n}(m, x) = ({}^{n}m, {}^{n}m)$. Therefore, we have ${}^{n}(m, x) = {}^{n-1}(m, x)$ since ${}^{n}m = {}^{n-1}m$.

Case 2. (m, x) ⪯ e. If ${}^{2}m < m$, we obtain $(m, x) \circ (m, x) = ({}^{2}m, {}^{2}m)$, $(m, x) \circ ({}^{2}m, {}^{2}m) = ({}^{3}m, {}^{3}m)$ and thus, as above, ${}^{n}(m, x) = {}^{n-1}(m, x)$. Otherwise, since ${}^{2}m = m \leq t$, we have (m, x) ○ (m, x) = (m, x) and thus ${}^{n-1}(m, x) = {}^{n}(m, x)$.

For $\mathbf{P^r_nMIAL}$, we have to further prove right *n*-potency, i.e., $(m, x)^{n} = (m, x)^{n-1}$ for $2 \leq n$ and for (m, x) ∈ X. The proof is analogous to that of left n-potency. This completes the proof. □

Proposition 3.4 Every countable linearly ordered L-algebra can be embedded into a standard algebra.

Proof: In an analogy to the proof of Proposition 3 in Yang (2016), we prove this. Let X, **A**, etc. be as in Proposition 3.3. Since (X, $\leq$) is a countable, dense, linearly-ordered set with maximum and minimum, it is order isomorphic to ($\mathbf{Q} \cap [0, 1], \leq$). Let g be such an isomorphism. If (I) to (V) in Proposition 3.3 hold, letting, for $\alpha, \beta \in [0, 1]$, $\alpha \circ' \beta = g(g^{-1}(\alpha) \circ g^{-1}(\beta))$, and, for all $m \in A$, $h'(m) = g(h(m))$, we obtain that $\mathbf{Q} \cap [0, 1], \leq, 1, 0, e, \partial, \circ', h'$ satisfy the conditions (I) to (V) of Proposition 3.3 whenever X, $\leq$, Max, Min, e, ∂, $\circ$, and h do. Thus, without loss of generality, we can assume that $X = \mathbf{Q} \cap [0, 1]$ and $\leq = \leq$, and so $\circ = \circ'$.

Now, we define for $\alpha, \beta \in [0, 1]$,

$$\alpha \circ'' \beta = \sup_{x \in X: x \leq \alpha} \sup_{y \in X: y \leq \beta} x \circ y.$$

The monotonicity and identity of $\circ''$ are easy consequences of the definition. Furthermore, it follows from the definition that $\circ''$ is conjunctive, i.e., $0 \circ'' 1 = 1 \circ'' 0 = 0$. For the left-continuity of $\circ''$, see Proposition 3 in Yang (2016).

We prove left and right n-potency properties. Suppose that $<\alpha_i : i \in N>$ is an increasing sequence of reals in [0, 1] such that

$\sup\{\alpha_i : i \in N\} = \alpha$. First note that ${}^{n-1}\alpha = \sup\{{}^{n-1}q : q \in \mathbf{Q} \cap [0, 1], q \leq \alpha\}$ and ${}^{n}\alpha = \sup\{{}^{n}q : q \in \mathbf{Q} \cap [0, 1], q \leq \alpha\}$. For the left *n*-potency of $\circ''$, we have to show ${}^{n-1}\alpha = {}^{n}\alpha$, $2 \leq n$. Since ${}^{n-1}q = {}^{n}q$, we have that $\sup\{{}^{n-1}q : q \in \mathbf{Q} \cap [0, 1], q \leq \alpha\} = \sup\{{}^{n}q : q \in \mathbf{Q} \cap [0, 1], q \leq \alpha\}$; therefore, ${}^{n-1}\alpha = {}^{n}\alpha$. The proof of the right *n*-potency of $\circ''$ is analogous.

It is an easy consequence of the definition that $\circ''$ extends $\circ$. By (I) to (V), *h* is an embedding of (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *) into ([0, 1], $\leq$, 1, 0, e, ∂, min, max, $\circ''$). Finally, for the fact that $\circ''$ has a residuated pair of implications, calling it ($\rightarrow''$, $\Rightarrow''$), see Proposition 3 in Yang (2016). □

Theorem 3.4 (Strong standard completeness) For L $\in$ Ls, the following are equivalent:

(1) T $\vdash_L \phi$.

(2) For every standard L-algebra and evaluation *v*, if $v(\psi) \geq e$ for all $\psi \in$ T, then $v(\phi) \geq e$.

Proof: The (1)-to-(2) direction is obvious. We prove the (2)-to-(1) direction. Let ϕ be a formula such that T $\nvdash_L \phi$, **A** a linearly ordered L-algebra, and v an evaluation in **A** such that v(ψ) $\geq$ t for all $\psi \in$ T and v(ϕ) < t. Let *h′* be the embedding of **A** into the standard L-algebra as in proof of Proposition 3.3. Then, $h' \circ$ v is an evaluation into the standard L-algebra such that $h' \circ$ v(ψ) $\geq e$ and yet $h' \circ$ v(ϕ) $< e$. □

4. Fixpointed involutive extensions

Here we provide standard completeness for the fixpointed involutive extensions of L $\in$ Ls.

First, we introduce two negations as defined connectives:

df2. $\neg\phi := \phi \rightarrow \mathbf{f}$, and
df3. $\sim\phi := \phi \Rightarrow \mathbf{f}$.

We next introduce some involutive extensions and their corresponding algebras.

Definition 4.1 (i) (Yang (2017a)) Involutive MIAL system **IMIAL** is the **MIAL** with the following axiom schemes:

A18. $\sim\neg\phi \rightarrow \phi$ (double negation elimination, DNE(1))

A19. $\neg\sim\phi \rightarrow \phi$ (double negation elimination, DNE(2))

(ii) (Yang (2017b)) For L $\in$ Ls, fixpointed and involutive extensions of L are introduced as follows:[12)]

- Right n-potent, fixpointed, involutive mianorm logic $\mathbf{P^r{}_nFIMIAL}$ is **IMIAL** plus ($P^r{}_n$) and (fixpoint, FP) $\mathbf{t} \leftrightarrow \mathbf{f}$.
- Left n-potent, fixpointed, involutive mianorm logic $\mathbf{P^l{}_nFIMIAL}$ is **IMIAL** plus ($P^l{}_n$) and (FP).

Definition 4.2 (i) Let $\neg x = x \rightarrow f$ and $\sim x = x \Rightarrow f$ for all

---

12) $\mathbf{P^r{}_nFIMIAL}$ and $\mathbf{P^l{}_nFIMIAL}$ were denoted by $\mathbf{IMIALc}^{lr}{}_n\mathrm{f}$ and $\mathbf{IMIALc}^{l}{}_n\mathrm{f}$, respectively, in Yang (2017b).

$x \in A$. An IMIAL-algebra is an MIAL-algebra satisfying: for all $x \in A$, (DNE(1)$^{\mathbf{A}}$) $\sim\neg x \leq x$ and (DNE(2)$^{\mathbf{A}}$) $\neg\sim x \leq x$.
(ii) A *$P^r{}_n$FIMIAL-algebra* is an IMIAL-algebra satisfying ($P^r{}_n{}^{\mathbf{A}}$) and ($f^{\mathbf{A}}$) t = f; a *$P^{rl}{}_n$FIMIAL-algebra* is an IMIAL-algebra satisfying ($P^l{}_n{}^{\mathbf{A}}$) and ($f^{\mathbf{A}}$). For convenience, we call these algebras FIL-algebras.

Now we consider standard completeness for FIL $\in$ {$\mathbf{P^r{}_nFIMIAL}$, $\mathbf{P^l{}_nFIMIAL}$}.

Fact 4.3 (Yang (2017a)) For every finite or countable linearly ordered **IMIAL**-algebra $\mathbf{A} = (A, \leq_A, \top, \bot, t, f, \wedge, \vee, *, \rightarrow, \Rightarrow)$, there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the conditions (I) to (IV) in Fact 3.1 and the following condition hold:
(VI) For all $x \in X$, x is involutive, i.e., it satisfies (DNE(1)$^{\mathbf{A}}$) and (DNE(2)$^{\mathbf{A}}$).

Proposition 4.4 For every finite or countable linearly ordered FIL-algebra $\mathbf{A} = (A, \leq_A, \top, \bot, t, f, \wedge, \vee, *, \rightarrow, \Rightarrow)$, there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the conditions (I) to (V) in Proposition 3.3, (VI) in Fact 4.3, and ($f^{\mathbf{A}}$).

Proof: We first note that, for **IMIAL**, m^+ denotes the successor of m if it exists, otherwise $m^+ = m$, for each $m \in A$. Note that, since the pair of negations in A, defined as $\neg m := m \rightarrow \partial$ and

$\sim m := m \Rightarrow \partial$, is involutive, we have that: $m = (\neg n)^{+}$ iff $n = (\neg m)^{+}$ and $m = (\sim n)^{+}$ iff $n = (\sim m)^{+}$; moreover, if $m < m^{+}$, then $(\neg(m^{+}))^{+} = \neg m$ and $(\sim(m^{+}))^{+} = \sim m$. Here, we use Y below in place of the X above. Let (Y, $\leq$) be the linearly ordered set, defined by

$$Y = \{(m, m): m \in A\} \cup \{(m, x): \exists m' \in A \text{ such that } m = m'^{+} > m', \text{ and } x \in Q \cap (0, m)\},$$

and $\leq$ being the corresponding lexicographic ordering as above. Then, it suffices to check the condition (V). In order to distinguish the two $\circ$'s introduced in Proposition 3.3, we denote the first and second circles as $\circ_{Y1}$ and $\circ_{Y2}$, respectively.

Now, we define new operations $\odot_{Y1}$ on Y, based on $\circ_{Y1}$, and $\odot_{Y2}$ on Y, based on $\circ_{Y2}$, as follows (but, for convenience, henceforth dropping each index if we need not distinguish them):[13)]

$$(m,x)\odot(n,y) = \begin{cases} \min\{\partial,(m,x)\circ(n,y)\} & \text{if } m=(\neg n)^{+} \text{ and } p/q+p'/q' \leq 1, \\ & \quad \text{where } x = mp/q \text{ and } y=np'/q', \\ & \text{or } m < (\neg n)^{+}; \text{ or} \\ & \quad \text{if } m=(\sim n)^{+} \text{ and } p/q+p'/q' \leq 1, \\ & \quad \text{where } x = mp/q \text{ and } y=np'/q', \\ & \text{or } m < (\sim n)^{+}; \text{ or} \\ (m,x) \circ (n,y) & \text{otherwise.} \end{cases}$$

13) This definition was introduced in Yang (2017a)

Note that $\odot_{Y1}$ is for $\mathbf{P^r_nFIMIAL}$ and $\mathbf{P^l_nFIMIAL}$, $3 \leq n$, and $\odot_{Y2}$ is for $\mathbf{P^r_2FIMIAL}$ (= $\mathbf{P^l_2FIMIAL}$).

The operation $\odot$ satisfies conditions (I) to (IV), (VI), and (f^A) (see Proposition 2 in Yang (2017a) and Proposition 3.2 in Yang (2017b)). Thus, we need to consider the condition (V).

We prove the left n-potency of $\odot$, i.e., $^n(m, x) = {}^{n-1}(m, x)$, $2 \leq n$.

Case 1. $m = (\sim m)^+$ and $2p/q \leq 1$, where $x = mp/q$, or $m < (\sim m)^+$.

Subcase 1.1. $m = {}^2m$. Since $t < m$ is not the case, we have $m = {}^2m \leq t = f < (\sim m)^+$ and thus $(m, x) \odot_{Y2} (m, x) = \min\{\partial, (m, x) \circ_{Y2} (m, x)\} = (m, x) \circ_{Y2} (m, x) = (m, x)$; therefore, $^n(m, x) = {}^{n-1}(m, x)$ since $^3m = {}^2m$ and thus $^nm = {}^{n-1}m$ for $2 < n$.

Subcase 1.2. $m \neq {}^2m$. We have to show $^n(m, x) = {}^{n-1}(m, x)$ for $2 < n$. Since the condition implies $^2m < m < t$, we have $(m, x) \odot_{Y1} (m, x) = \min\{\partial, (m, x) \circ_{Y1} (m, x)\} = (m, x) \circ_{Y1} (m, x)$. Therefore, as above, we have that $^n(m, x) = {}^{n-1}(m, x)$.

Case 2. $m = (\neg m)^+$ and $2p/q \leq 1$, where $x = mp/q$, or $m < (\neg m)^+$. The proof is analogous to that of Case 1.

Case 3. Otherwise. The proof is reducible to that of the left *n*-potency for $\mathbf{P^l_nMIAL}$ in Proposition 3.3.

The proof for the right *n*-potency of $\odot$, i.e., $(m, x)^n = (m, x)^{n-1}$, $2 \leq n$, is analogous to the left one. □

Proposition 4.5 Every countable linearly ordered FIL-algebra can be embedded into a standard algebra.

Proof: The proof of this claim is analogous to that of Proposition 3.4. □

Theorem 4.6 (Strong standard completeness) For FIL ∈ { $\mathbf{P^r{}_nFIMIAL}$, $\mathbf{P^l{}_nFIMIAL}$}, the following are equivalent:

(1) T $\vdash_{FIL}$ ϕ.

(2) For every standard **FIL**-algebra and evaluation v, if v(ψ) ≥ *e* for all ψ ∈ T, then v(ϕ) ≥ *e* .

Proof: The (1)-to-(2) direction follows from the definition. For the (2)-to-(1) direction, let ϕ be a formula such that T $\nvdash_{FIL}$ ϕ, **A** a linearly ordered **FIL**-algebra, and v an evaluation in **A** such that v(ψ) ≥ t for all ψ ∈ T and v(ϕ) < t. Let *h′* be the embedding of **A** into the standard **FIL**-algebra as in proof of Proposition 4.4. Then, *h′* ⊙ v is an evaluation into the standard **FIL**-algebra such that *h′* ⊙ v(ψ) ≥ *e* and yet *h′* ⊙ v(ϕ) < *e* . □

5. Concluding remark

We investigated (not merely algebraic completeness for $\mathbf{P^r{}_nMIAL}$ and $\mathbf{P^l{}_nMIAL}$ but also) standard completeness for $\mathbf{P^r{}_nMIAL}$ and $\mathbf{P^l{}_nMIA}$ via Yang's construction in the style of Jenei−Montagna. We further considered their fixpointed involutive extensions. Note that this construction does not work for their involutive extensions (see Yang (2017b)). To introduce such semantics for their involutive extensions is a problem left in this paper.

References

Baldi, P. (2014), “A note on standard completeness for some extensions of uninorm logic”, *Soft Computing*, 18, pp. 1463-1470.

Ciabattoni, A., Esteva, F., and Godo, L. (2002), “T-norm based logics with *n*-contraction”, *Neural Network World*, 12, pp. 441-453.

Cintula, P. (2006), “Weakly Implicative (Fuzzy) Logics I: Basic properties”, *Archive for Mathematical Logic*, 45, pp. 673-704.

Cintula, P., Horĉík, R., and Noguera, C. (2013), “Non-associative substructural logics and their semilinear extensions: axiomatization and completeness properties”, *Review of Symbol. Logic*, 12, pp. 394-423.

Cintula, P., Horĉík, R., and Noguera, C. (2015), “The quest for the basic fuzzy logic”, *Mathematical Fuzzy Logic*, P. Hájek (Ed.), Springer.

Cintula, P. and Noguera, C. (2011), A general framework for mathematical fuzzy logic, *Handbook of Mathematical Fuzzy Logic*, vol 1, P. Cintula, P. Hájek, and C. Noguera (Eds.), London, College publications, pp. 103-207.

Esteva. F. and Godo. L. (2001), “Monoidal t-norm based logic: towards a logic for left-continuous t-norms”, 124, pp. 271-288.

Galatos, N., Jipsen, P., Kowalski, T., and Ono, H. (2007), *Residuated lattices: an algebraic glimpse at substructural logics*, Amsterdam, Elsevier.

Hájek, P. (1998), *Metamathematics of Fuzzy Logic*, Amsterdam, Kluwer.

Horĉík, R. (2011), Algebraic semantics: semilinear FL-algebras, *Handbook of Mathematical Fuzzy Logic*, vol 1, P. Cintula, P. Hájek, and C. Noguera (Eds.), London, College publications, pp. 283-353.

Hori, R., Ono, H., and Schellinx, H. (1994), "Extending intuitionistic linear logic with knotted structural rules", *Studia Logica*, 35, pp. 219-2424.

Jenei, S. and Montagna, F. (2002), "A Proof of Standard completeness for Esteva and Godo's Logic MTL", *Studia Logica*, 70, pp. 183-192.

Metcalfe, G., and Montagna, F. (2007), "Substructural Fuzzy Logics", *Journal of Symbolic Logic*, 72, pp. 834-864.

Wang, S. (2012), "Uninorm logic with the n-potency axiom", *Fuzzy Sets and Systems*, 205, pp. 116-126.

Yang, E. (2015), "Weakening-free, non-associative fuzzy logics: Micanorm-based logics", *Fuzzy Sets and Systems*, 276, pp. 43-58.

Yang, E. (2016), "Basic substructural core fuzzy logics and their extensions: Mianorm-based logics", *Fuzzy Sets and Systems*, 301, pp. 1-18.

Yang, E. (2017a), "Involutive basic substructural core fuzzy logics: Involutive mianorm-based logic", *Fuzzy Sets and Systems*, 320, pp. 1-16.

Yang, E. (2017b), "Some axiomatic extensions of the involutive mianorm logic **IMIAL**", *Korean Journal of Logic*, 20/3, pp. 313-333.

Yang, E. (2019), "Mianorm-based logics with *n*-contraction and *n*-mingle axioms", *Journal of Intelligent and Fuzzy systems*, 37, pp. 7895-7907.

좌, 우 *n*-멱등 공리를 갖는 미아놈 논리

양 은 석

이 글에서 우리는 좌, 우 *n*-멱등 공리를 갖는 미아놈에 기반한 논리를 다룬다. 이를 위하여 먼저 미아놈에 바탕을 둔 좌, 우 *n*-멱등 공리를 갖는 논리 체계 $\mathbf{P^r_nMIAL}$, $\mathbf{P^l_nMIAL}$을 소개한다. 각 체계에 상응하는 대수적 구조를 정의한 후, 이들 체계가 대수적으로 완전하다는 것을 보인다. 다음으로, 이 논리 체계들이 표준적으로 완전하다는 것 즉 단위 실수 [0, 1에서 완전하다는 것을 제네이-몬테그나 방식의 구성을 사용하여 보인다. 마지막으로 이를 고정점을 갖는 누승적 확장에 대한 연구로 확대한다.

주요어: 퍼지 논리, 미아놈, 대수적 완전성, 표준 완전성, *n*-멱등.