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Involutive Micanorm Logics with the n-potency axiom*

Eunsuk Yang

【Abstract】In this paper, we deal with some axiomatic extensions of the involutive micanorm logic **IMICAL**. More precisely, first, the two involutive micanorm-based logics **P$_n$IMICAL** and **FP$_n$IMICAL** are introduced. Their algebraic structures are then defined, and their corresponding algebraic completeness is established. Next, standard completeness is established for **FP$_n$IMICAL** using construction in the style of Jenei-Montagna.

【Key Words】fuzzy logic, involution, micanorm, algebraic completeness, standard completeness, **IMCAL**, fixed-point.

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1. Introduction

Metcalfe and Montagna (2007) introduced the weakening-free fuzzy logics **UL** (Uninorm logic), **IUL** (Involutive uninorm logic), **UML** (Uninorm mingle logic), and **IUML** (Involutive uninorm mingle logic) as substructural fuzzy logics based on uninorms[1]), and established standard completeness, i.e., completeness with respect to (w.r.t.) the corresponding unit interval structures, for them (except **IUL**[2])). One interesting fact is that the system **IUML** is not the system **UML** with the involution axiom $\sim\sim\phi \rightarrow \phi$. This system further requires the fixed-point axiom (F) $\mathbf{t} \leftrightarrow \mathbf{f}$. This makes us to think that some involutive fuzzy logics require the axiom (F) for their standard completeness. This idea is very natural in the sense that the standard negation 1 - x has the fixed-point 1/2, i.e., $1/2 = \sim(1/2)$.

The purpose of this paper is to verify the idea that some involutive non-associative basic fuzzy logics require that axiom for their standard completeness. As its simple example, we introduce one system without the fixed-point axiom and its extension having that axiom and establish standard completeness for the second system.

Before introducing the systems, we note some facts associated with those systems. The present author introduced *micanorms* (binary monotonic identity commutative aggregation operations on

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1) Uninmorms are functions introduced by Yager and Rybalov (1996) as a generalization of t-norms where the identity can lie anywhere in [0, 1].

2) For the proof of standard completeness for **IUL**, see Wang (201+).

the real unit interval [0, 1]) and logics based on micanorms and provided standard completeness for involutive such logics, which was a problem left open in Horčík (2011), using the Jenei-Montagna-style construction introduced in Esteva et al. (2002) and Jenei & Montagna (2002). After providing such completeness, he stated as follows:

> Wang defined a new monoid ⊙ based on Wang's monoid $\circ_W$ for involution and provided standard completeness for **CnIUL** in Wang (2013). Since Yang's monoid $\circ_{Y''}$ is also Wang's monoid, we can also define such a monoid based on $\circ_{Y''}$ and provide standard completeness results for **CnIUL** and similarly for **IMICAL** and **CnIMICAL**(Yang (2015a), p. 57).

Let ϕ^n stand for $((\cdots(\phi \,\&\, \phi) \,\&\, \cdots) \,\&\, \phi) \,\&\, \phi$, n ϕ's. The system **CnIMICAL** is the involutive micanorm logic **IMICAL** with (*n*-potency, nP) $\phi^n \leftrightarrow \phi^{n-1}$, $2 \leq n$.[3] As the statements in Remark 3 of Yang (2015a) show, although the author insists that the standard completeness using the construction in the style of Jenei-Montagna (the proof in Theorem 5) is applicable to **CnIMICAL**, its proof is not provided.

In an another paper (Yang (2015b)), the present author claimed that I verified that the proof in Theorem 5 of Yang (2015a) is applicable to **FCnIMICAL** (**CnIMICAL** with (F)) but not to **CnIMICAL**. However, that verification is not correct in the sense that the system considered in Yang (2015b) is not the real **CnIMICAL** in the sense that in place of (nP) the present author

3) For the important features of n-potency in logic and algebra, see Ciabattoni et al (2002), Wang (2012; 2013), and Kowalski (2004) as examples.

introduced (n-mingle, nM) $\phi^n \to \phi^{n-1}$, $2 \leq n$, as the n-potency axiom. Namely, the author introduced **CnIMICAL**. as **IMICAL** with (n-mingle, nM).

In order to reconsider the above claim, here we take the system **IMICAL** with (nP) and its extension with (F). This will satisfy both our purpose and the claim in Yang (2015b). Here we describe these two systems as **P_nIMIAL** and **FP_nIMIAL** in place of **CnIMICAL** and **FCnIMICAL** because the expression "Cn" in these names reminds us n-contraction in place of n-potency.

The paper is organized as follows. In Section 2, we present the axiomatizations of the systems **P_nIMICAL** and **FP_nIMICAL**, define their corresponding algebraic structures, by subvarieties of the variety of residuated lattices, and show that they are complete w.r.t. linearly ordered corresponding algebras. In Section 3, we establish standard completeness for the system **FP_nIMICAL**. using the method introduced in Yang (2015a; 2015b) together with the remark that this approach does not work for **P_nIMICAL**.

For convenience, we shall adopt notations and terminology similar to those in Cintula (2006), Esteva et al. (2002), Hájek (1998), Metcalfe & Montagna (2007), Yang (2009; 2013; 2014; 2015a; 2015b), and assume familiarity with them (together with the results found therein).

2. Syntax

We base some axiomatic extensions of the involutive micanorm logic **IMICAL** on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$, and
df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define ϕ^n_t as ϕ_t & $\cdots$ & ϕ_t, n factors, where $\phi_t := \phi \wedge \mathbf{t}$. For the rest of this paper, we use the customary notations and terminology, and the axiom systems to provide a consequence relation.

We start with the following axiomatization of **IMICAL**, the most basic fuzzy logic introduced here.

Definition 2.1 (Yang (2015a)) **IMICAL** consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)
A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ ($\wedge$-elimination, $\wedge$-E)
A3. $((\phi\rightarrow\psi)\wedge(\phi\rightarrow\chi)) \rightarrow (\phi\rightarrow(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)
A4. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)
A5. $((\phi\rightarrow\chi)\wedge(\psi\rightarrow\chi)) \rightarrow ((\phi\vee\psi)\rightarrow\chi)$ ($\vee$-elimination, $\vee$-E)
A6. $\mathbf{F} \rightarrow \phi$ (ex falso quadlibet, EF)
A7. $(\phi \,\&\, \psi) \rightarrow (\psi \,\&\, \phi)$ (&-commutativity, &-C)
A8. $(\mathbf{t} \rightarrow \phi) \leftrightarrow \phi$ (push and pop, PP)
A9. $\phi \rightarrow (\psi \rightarrow (\psi \,\&\, \phi))$ (&-adjunction, &-Adj)
A10. $(\phi_t \,\&\, \psi_t) \rightarrow (\phi \wedge \psi)$ (&$\wedge$)
A11. $(\psi \,\&\, (\phi \,\&\, (\phi \rightarrow (\psi \rightarrow \chi)))) \rightarrow \chi$ (residuation, Res')

A12. $((\phi \to (\phi \,\&\, (\phi \to \psi))) \,\&\, (\psi \to \chi)) \to (\phi \to \chi)$ (T')

A13. $((\delta \,\&\, \varepsilon) \to (\delta \,\&\, (\varepsilon \,\&\, (\phi \to \psi)_t))) \vee (\delta' \to (\varepsilon' \to ((\varepsilon' \,\&\, \delta') \,\&\, (\psi \to \phi)_t)))$ (PL)

A14. $\sim\sim\phi \to \phi$ (double negation elimination, DNE)

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi \vdash \phi_t$ (adj_u)

$\phi \vdash (\delta \,\&\, \varepsilon) \to (\delta \,\&\, (\varepsilon \,\&\, \phi))$ (α)

$\phi \vdash \delta \to (\varepsilon \to ((\varepsilon \,\&\, \delta) \,\&\, \phi))$ (β)

Definition 2.2 A logic is an axiomatic extension (extension for short) of an arbitrary logic **L** if and only if (iff) it results from **L** by adding axiom schemes. Especially, we introduce two particular extensions of **IMIAL**.

- N-potent involutive micanorm logic
  P$_n$IMICAL is **IMICAL** plus (nP) $\phi^n \leftrightarrow \phi^{n-1}$, $2 \leq n$.
- Fixed-pointed n-potent involutive micanorm logic
  FP$_n$IMICAL is **P$_n$IMICAL** plus (F) $\mathbf{t} \leftrightarrow \mathbf{f}$.

For easy reference, we let Ls be the set of the weakening-free, non-associative fuzzy logics defined in Definition 2.

Definition 2.3 Ls = {**P$_n$IMICAL**, **FP$_n$IMICAL**}

A *theory* over L ($\in$ Ls) is a set T of formulas. A *proof* in a theory over L is a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using a rule of L. T $\vdash$ φ, more exactly T $\vdash_L$ φ, means that φ is *provable* in T w.r.t. L, i.e., there is an L-proof of φ in T. A theory T is *inconsistent* if T $\vdash$ **F**; otherwise it is *consistent*.

The deduction theorem for L is as follows:

Proposition 2.4 (Cintula et al. (2013; 2015)) Let T be a theory, and φ, ψ formulas. T ∪ {φ} $\vdash_L$ ψ iff T $\vdash_L$ ɣ(φ) → ψ for some ɣ ∈ Π(bDT*).[4]

For convenience, "∼," "∧," "∨," and "→" are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

Suitable algebraic structures for L (∈ Ls) are obtained as a subvariety of the variety of commutative monoidal residuated lattices.

Definition 2.5 (Yang (2015a)) (i) A *pointed bounded commutative residuated lattice* is a structure **A** = (A, ⊤, ⊥, t, f, ∧, ∨, *, →) such that:

(Ⅰ) (A, ⊤, ⊥, ∧, ∨) is a bounded lattice with top element ⊤ and bottom element ⊥.

(Ⅱ) (A, *, t) is a commutative monoid.

(Ⅲ) y ≤ x→z iff x * y ≤ z, for all x, y, z ∈ A (residuation).

(ii) An *IMICAL-algebra* is a pointed bounded commutative

4) For ɣ and Π(bDT*), see Cintula et al. (2013; 2015) and Yang (2015a).

residuated lattice satisfying

- $t \leq ((z*w) \rightarrow (z*(w*(x \rightarrow y)t))) \vee (z' \rightarrow (w' \rightarrow ((w'*z')*(y \rightarrow x)t)))$, for all x, y, z, w, z', w' $\in$ A (PL^{A}).
- $t \leq \sim\sim x \rightarrow x$, for all x $\in$ A (DNE^{A}).

L-algebras the class of which characterizes L are defined as follows.

Definition 2.6 (L-algebras) The algebraic (in)equations corresponding to the structural axioms introduced in Definition 2.2 are defined as follows: for all x $\in$ A,

- $x^{n} = x^{n-1}$, $2 \leq n$, (nP^{A})
- $t = f$ (F^{A}).

A *P_nMICAL-algebra* is an IMICAL-algebra satisfying (nP^{A}) and a *FP_nIMICAL-algebra* is a P_nIMICAL-algebra satisfying (F^{A}). We call these algebras *L-algebras*.

An L-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 2.7 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function v : FOR $\rightarrow \mathcal{A}$ satisfying: $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{F}) = \bot$, $v(\mathbf{f}) = f$, (and hence $v(\sim\phi) = \sim v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = t$).

Definition 2.8 Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and K a class of L-algebras.

(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an $\mathcal{A}$-*tautology* (or $\mathcal{A}$-*valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-*model* of T if $v(\phi) \geq t$ for each $\phi \in T$. We denote the class of $\mathcal{A}$-models of T, by *Mod(T,* $\mathcal{A}$*)*.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_{\mathsf{K}} \phi$, if Mod(T, $\mathcal{A}$) = Mod($T \cup \{\phi\}$, $\mathcal{A}$) for each $\mathcal{A} \in \mathsf{K}$.

Definition 2.9 (**L**-algebra, Cintula (2006)) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is an ***L****-algebra* iff, whenever ϕ is L-provable in T (i.e. $T \vdash_L \phi$, L an L logic), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. $T \vDash_{\{A\}} \phi$), $\mathcal{A}$ a corresponding L-algebra. By $MOD^{(l)}(L)$, we denote the class of (linearly ordered) **L**-algebras. Finally, we write $T \vDash^{(l)}_L \phi$ in place of $T \vDash_{MOD^{(l)}(L)} \phi$.

Theorem 2.10 (Strong completeness) Let T be a theory, and ϕ a formula. $T \vdash_L \phi$ iff $T \vDash_L \phi$ iff $T \vDash^1_L \phi$.

Proof: We obtain this theorem as a corollary of Theorem 3.1.8 in Cintula & Noguera (2011). □

3. Standard completeness

In this section, we provide standard completeness results for **FP_nIMICAL** using the Jenei-Montagna-style construction in Eeteva et al. (2002) and Jenei & Montagna (2002).

We first show that finite or countable, linearly ordered **FP_nIMICAL**-algebras are embeddable into a standard algebra. (For convenience, we add the 'less than or equal to' relation symbol "≤" to such algebras.) First, note the following results.

Theorem 3.1 (Yang (2015a))

(i) For every finite or countable linearly ordered **MICAL**-algebra **A** = (A, $\leq_A$, ⊤, ⊥, t, f, ∧, ∨, *, →), there is a countable ordered set X, a binary operation ○, and a map h from A into X such that the following conditions hold:

(Ⅰ) X is densely ordered, and has a maximum Max, a minimum Min, and special elements e, ∂.

(Ⅱ) (X, ○, ≤, e) is a linearly ordered, monotonic, commutative groupoid with unit.

(Ⅲ) ○ is conjunctive and left-continuous w.r.t. the order topology on (X, ≤).

(Ⅳ) h is an embedding of the structure (A, $\leq_A$, ⊤, ⊥, t, f, ∧, ∨, *) into (X, ≤, Max, Min, e, ∂, min, max, ○), and for all m, n ∈ A, h(m → n) is the residuum of h(m) and h(n) in (X, ≤, Max, Min, e, ∂, max, min, ○).

(ii) For every finite or countable linearly ordered **IMICAL**-algebra **A** = (A, $\leq_A$, ⊤, ⊥, t, f, ∧, ∨, *, →), there is a countable ordered set X, a binary operation ○, and a map h from A into X such that the conditions (I) to (IV) in (i) and the following condition hold:

(V) For all x ∈ X, x is involutive, i.e., it satisfies (DNE^{A}).

Proposition 3.2 For every finite or countable linearly ordered **FP_nIMICAL**-algebra **A** = (A, $\leq_A$, ⊤, ⊥, t, f, ∧, ∨, *, →), there is a countable ordered set X, a binary operation ○, and a map h from A into X such that the conditions (I) to (V) of (ii) in Theorem 3.1 and the following condition hold:

(A) (X, ○, ≤, e) is n-potent and fixed-pointed.

Proof: For convenience, we assume A as a subset of **Q** ∩ [0, 1] with a finite or countable number of elements, where 0 and 1 are least and greatest elements, respectively, each of which corresponds to ⊤ and ⊥, respectively.

We first note that, for **MICAL**, a linearly ordered, monotonic groupoid with unit (X, ○, ≤, e) is defined as follows:

$$X = \{(m, x): m \in A \setminus \{0\ (= \bot)\} \text{ and } x \in \mathbf{Q} \cap (0, m]\} \cup \{(0, 0)\};$$

for (m, x), (n, y) ∈ X,

(m, x) ≤ (n, y) iff either m $<_A$ n, or m $=_A$ n and x ≤ y;

(m,x) ○ (n,y) = max{(m,x), (n,y)} if m*n $=_A$ m∨n, m $\neq_A$ n, and

$$\begin{aligned}
&\qquad\qquad\quad (m, x) \leq e \text{ or } (n, y) \leq e;\\
&\min\{(m,x), (n,y)\} \text{ if } m * n = m \wedge z, \text{ and}\\
&\qquad\qquad\quad (m, x) \leq e \text{ or } (n, y) \leq e;\\
&(m * n, m * n) \quad \text{otherwise.}
\end{aligned}$$

For convenience, we henceforth drop the index A in $\leq_A$ and $=_A$, if we need not distinguish them. Context should clarify the intention.

We next note that, for **IMICAL**, m^+ denotes the successor of m if it exists, otherwise $m^+ = m$, for each $m \in A$; since the negation in A, defined as $\sim m := m \rightarrow f$ is involutive, we have that: $m = (\sim n)^+$ iff $n = (\sim m)^+$; moreover, if $m < m^+$, then $(\sim(m^+))^+ = \sim m$. Here, we use Y below in place of the X above. Let $(Y, \leq)$ be the linearly ordered set, defined by

$$Y = \{(m, m): m \in A\} \cup$$
$$\{(m, x): \exists m' \in A \text{ such that } m = m'^+ > m', \text{ and } x \in Q \cap (0, m)\},$$

and $\leq$ being the corresponding lexicographic ordering as above. It is clear that $(Y, \leq)$ is a subset of the ordered set $(X, \leq)$ defined as above with the same bounds and special elements e (= (t, t)) and ∂ (= (f, f)). Notice that Y is closed under $\circ$ and that $\leq$ is a linear order with maximum (1, 1), minimum (0, 0), and special elements e and ∂. Furthermore, $\leq$ is dense. This proves (I).

For condition (II), we need to define a new operation $\odot$ on Y, based on $\circ$, as follows:

$(m,x)\odot(n,y) = \min\{\partial,(m,x)\circ(n,y)\}$ if $m = (\sim n)^+$ and $p/q+p'/q'\leq 1$,
where $x = mp/q$ and $y=np'/q'$,
or $m < (\sim n)^+$;
$(m,x) \circ (n,y)$ otherwise.

The operation $\odot$ satisfies conditions (II) to (V) (see Theorem 5 in Yang (2015a)).

Now we note that for **FP$_n$IMICAL**, $3 \leq n$, the groupoid operation $\odot$ is defined based on the definition of $\circ$ above, whereas for **FP$_2$IMICAL** the groupoid operation $\odot$ is defined based on the following definition of $\circ$: for $(m, x), (n, y) \in X$,

$(m,x) \circ (n,y) = \max\{(m,x), (n,y)\}$ if $m * n =_A m \vee n$ and
$(m, x) \succ e$ or $(n, y) \succ e$;
$\min\{(m,x), (n,y)\}$ if $m * n = m \wedge z$, and
$(m, x) \leq e$ or $(n, y) \leq e$;
$(m * n, m * n)$ otherwise.

The proof for **FP$_n$IMICAL** is analogous to that for **IMIAL**. For **FP$_n$IMICAL**, we need to prove that $(X, \odot, \leq, e)$ satisfies the condition (A), i.e., ($nP^{\mathbf{A}}$) and ($F^{\mathbf{A}}$). We first prove the n-potency of $\odot$, i.e., $(m, x)^n = (m, x)^{n-1}$, $2 \leq n$.

Case 1. $m = (\sim m)^+$ and $2p/q \leq 1$, where $x = mp/q$, or $m < (\sim m)^+$.

Subcase 1.1. $m^2 = m$. Since $t < m$ is not the case, we have $m = m^2 \leq t = f < (\sim m)^+$ and thus $(m, x) \odot (m, x) = \min\{\partial, (m, x) \circ (m, x)\} = (m, x) \circ (m, x) = (m, x)$; therefore, $(m, x)^n =$

$(m, x)^{n-1}$ since $m^3 = m^2$ and thus $m^n = m^{n-1}$ for $2 \leq n$.

Subcase 1.2. $m \neq m^2$. We need to show $(m, x)^n = (m, x)^{n-1}$ for $2 < n$. Since the condition implies $m^2 < m < t$, we have $(m, x) \odot (m, x) = \min\{\partial, (m, x) \circ (m, x)\} = (m, x) \circ (m, x) = = (m^2, m^2)$, $(m^2, m^2) \odot (m, x) = (m^2, m^2) \circ (m, x) = (m^3, m^3)$ and thus $(m, x)^{n-1} = (m^{n-1}, m^{n-1})$ and $(m, x)^n = (m^n, m^n)$. Therefore, we have $(m, x)^n = (m, x)^{n-1}$ since $m^n = m^{n-1}$.

Case 2. Otherwise. The condition implies $e \prec (m, x)$. If $m^2 = m$, then $(m, x) \circ (m, x) = (m, x)$ and thus $(m, x)^n = (m, x)^{n-1}$. Otherwise, we have $t < m < m^2$ and thus $(m, x) \circ (m, x) = (m^2, m^2)$. Hence, as above, we also have $(m^2, m^2) \circ (m, x) = (m^3, m^3)$ and thus $(m, x)^n = (m, x)^{n-1}$ since $m^n = m^{n-1}$.

The proof of fixed-point is easy since $t = f$ and thus $e = (t, t) = (f, f) = \partial$. □

Proposition 3.3 Every countable linearly ordered **FP$_n$IMICAL**-algebra can be embedded into a standard algebra.

Proof: In an analogy to the proof of Theorem 3.2 in Jenei & Montagna (2002), we prove this. Let X, **A**, etc. be as in Proposition 3.2. Since $(X, \leq)$ is a countable, dense, linearly-ordered set with maximum and minimum, it is order isomorphic to $(\mathbf{Q} \cap [0, 1], \leq)$. Let g be such an isomorphism. If (I) to (V) and (A) hold, letting for $\alpha, \beta \in [0, 1]$, $\alpha \odot' \beta = g(g^{-1}(\alpha) \odot g^{-1}(\beta))$, and, for all $m \in A$, $h'(m) = g(h(m))$, we obtain that $\mathbf{Q} \cap [0, 1]$, $\leq$, 1, 0, e, ∂, $\odot'$, h' satisfy the conditions (I) to (V) and (A) of Proposition 3.2 whenever X, $\leq$, Max, Min, e, ∂, $\odot$, and h do. Thus, without loss of generality, we can assume that X = $\mathbf{Q} \cap [0, 1]$ and $\preceq\ = \ \leq$.

Now we define for $\alpha, \beta \in [0, 1]$,

$$\alpha \odot'' \beta = \sup_{x \in X: x \leq \alpha} \sup_{y \in X: y \leq \beta} x \odot y.$$

Commutativity of $\odot''$ follows from that of $\odot$. Its monotonicity, identity, fixed-point, and *n*-potency are easy consequences of the definition. Furthermore, it follows from the definition that $\odot''$ is conjunctive, i.e., $0 \odot'' 1 = 0$.

We prove left-continuity. Suppose that $<\alpha_n: n \in \mathbf{N}>$, $<\beta_n: n \in \mathbf{N}>$ are increasing sequences of reals in [0, 1] such that $\sup\{\alpha_n: n \in \mathbf{N}\} = \alpha$ and $\sup\{\beta_n: n \in \mathbf{N}\} = \beta$. By the monotonicity of $\odot''$, $\sup\{\alpha_n \odot'' \beta_n\} = \alpha \odot'' \beta$. Since the restriction of $\odot''$ to $\mathbf{Q} \cap [0, 1]$ is left-continuous, we obtain

$$\begin{aligned} \alpha \odot'' \beta &= \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q \leq \alpha, r \leq \beta\} \\ &= \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\}. \end{aligned}$$

For each $q < \alpha$, $r < \beta$, there is n such that $q < \alpha_n$ and $r < \beta_n$. Thus,

$$\sup\{\alpha_n \odot'' \beta_n: n \in \mathbf{N}\} \geq \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\} = \alpha \odot'' \beta.$$

Hence, $\odot''$ is a left-continuous involutive micanorm on [0, 1]. It is an easy consequence of the definition that $\odot''$ extends

$\odot$. By (I) to (V) and (A), h is an embedding of (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *) into ([0, 1], $\leq$, 1, 0, e, ∂, min, max, $\odot''$). Moreover, $\odot''$ has a residuum, calling it $\rightharpoonup$.

We finally prove that for x, y $\in$ A, h(x $\rightarrow$ y) = h(x) $\rightharpoonup$ h(y). By (IV), h(x $\rightarrow$ y) is the residuum of h(x) and h(y) in (**Q** $\cap$ [0, 1], $\leq$, 1, 0, e, ∂, min, max, $\odot$). Thus

$$h(x) \odot'' h(x \rightarrow y) = h(x) \odot h(x \rightarrow y) \leq h(y).$$

Suppose toward contradiction that there is α > h(x $\rightarrow$ y) such that α $\odot''$ h(x) $\leq$ h(y). Since **Q** $\cap$ [0, 1] is dense in [0, 1], there is q $\in$ **Q** $\cap$ [0, 1] such that h(x $\rightarrow$ y) < q $\leq$ α. Hence q $\odot''$ h(x) = q $\odot$ h(x) $\leq$ h(y), contradicting (IV). □

Theorem 3.4 (Strong standard completeness) For **FP$_n$IMICAL**, the following are equivalent:

(1) T $\vdash_{\mathbf{FPnIMICAL}}$ ϕ.

(2) For every standard **FP$_n$IMICAL**-algebra and evaluation v, if v(ψ) $\geq$ e for all ψ $\in$ T, then v(ϕ) $\geq$ e.

Proof: (1) to (2) follows from definition. We prove (2) to (1). Let ϕ be a formula such that T $\nvdash_{\mathbf{FPnIMICAL}}$ ϕ, **A** a linearly ordered **FP$_n$IMICAL**-algebra, and v an evaluation in **A** such that v(ψ) $\geq$ t for all ψ $\in$ T and v(ϕ) < t. Let h′ be the embedding of **A** into the standard **L**-algebra as in proof of Proposition 3.3. Then, h′ $\odot$ v is an evaluation into the standard **FP$_n$IMICAL**-algebra such that h′ $\odot$ v(ψ) $\geq$ e and yet h′ $\odot$ v

$(\phi) < e$. □

Remark 3.5 The proof of standard completeness in Theorem 3.4 does not work for **P$_n$IMICAL** because the definition of $\odot$ does not satisfy the *n*-potency property. Consider the following case: $0 < f < m$, $\sim m < (\sim m)^+ < t < 1$. Let $m = m * m$, we have $(m, x) \odot (m, x) = \min\{\partial, (m, x) \circ (m, x)\} = \partial \prec (m, x)$; therefore, $(m, x) \neq (m, x) \odot (m, x)$. Otherwise, let $(f * m) * f < f * m < m$. We have $(m, x)^3 = \partial \odot (m, x) = (f * m, f * m) \neq (m, x)^2$. Therefore, we have $(m, x)^n \neq (m, x)^{n-1}$ for $2 \leq n$.

4. Concluding remark

We investigated (not merely algebraic completeness for **P$_n$IMICAL** and **FP$_n$IMICAL** but also) standard completeness for **FP$_n$IMICAL**. We further noted that the proof of standard completeness does not work for **P$_n$IMICAL**.

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전북대학교 철학과, 비판적사고와논술연구소
Department of Philosophy & Institute of Critical Thinking and Writing, Chonbuk National University
eunsyang@jbnu.ac.kr

N-멱등 공리를 갖는 누승적 미카놈 논리

양 은 석

이 글에서 우리는 누승적 미카놈 논리 **IMICAL**의 몇몇 공리적 확장 체계를 다룬다. 보다 구체적으로, 먼저 누승적 미아놈에 바탕을 두 논리 체계 $\mathbf{P_nIMIAL}$, $\mathbf{FP_nIMIAL}$을 소개한다. 각 체계에 상응하는 대수적 구조를 정의한 후, 이들 체계가 대수적으로 완전하다는 것을 보인다. 다음으로, 이 논리 체계들 중 $\mathbf{FP_nIMIAL}$가 표준적으로 완전하다는 것 즉 단위 실수 [0,1]에서 완전하다는 것을 제네이-몬테그나 방식의 구성을 사용하여 보인다.

주요어: 퍼지 논리, 누승, 미카놈, 대수적 완전성, 표준 완전성, **IMICAL**, 고정점