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Weakly associative fuzzy logics*

Eunsuk Yang

【Abstract】This paper investigates weakening-free fuzzy logics with three weak forms of associativity (of multiplicative conjunction &). First, the wta-uninorm (based) logic $\mathbf{WA_tMUL}$ and its two axiomatic extensions are introduced as weakening-free weakly associative fuzzy logics. The algebraic structures corresponding to the systems are then defined, and algebraic completeness results for them are provided. Next, standard completeness is established for $\mathbf{WA_tMUL}$ and the two axiomatic extensions with an additional axiom using construction in the style of Jenei-Montagna.

【Key Words】weakening-free fuzzy logic, weak associativity, algebraic completeness, standard completeness, uninorm.

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1. Introduction

The purpose of this paper is to introduce weakening-free fuzzy logics assuming weak forms of associativity but not associativity (of multiplicative conjunction &). For this, recall first historical facts associated with fuzzy logics based on aggregation operators generalizing t-norms. Metcalfe (and Montagna) (2004; 2007) introduced the weakening-free fuzzy logics **UL** (Uninorm logic), **IUL** (Involutive uninorm logic), **UML** (Uninorm mingle logic), and **IUML** (Involutive uninorm mingle logic) as substructural fuzzy logics, and established standard completeness, i.e., completeness with respect to (w.r.t.) the corresponding unit interval structures, for them (except **IUL**). These logics are based on *uninorms*, which are functions introduced by Yager and Rybalov (1996) as a generalization of t-norms where the identity can lie anywhere in [0, 1]. Before introducing uninorms, Yager introduced a generalization of uninorms, a variant of the concept of uninorm obtained by removing the associativity condition in its definition: in Yager (1994a; 1994b), he introduced a class of MICA (Monotonic Identity Commutative Aggregation) operators together with the mention that MICA operators constitute the basic operators needed for aggregation in fuzzy system modeling. Yang (2015) recently defined micanorms as binary MICA operations, and introduced several weakening-free fuzzy logics based on micanorms.

The weakening-free fuzzy systems introduced in Yang (2015) are non-associative fuzzy logics in that they lack associativity (of

&). In this paper we investigates weakening-free fuzzy logics with three weak forms of associativity (of &). Let us call such logics *weakening-free weakly associative fuzzy logics*. The reason to investigate these logics is related to the followings: as Metcalfe and Montagna mentioned in Metcalfe & Montagna (2007), establishing standard completeness of fuzzy logics proves to be more challenging. One method introduced in Jenei & Montagna (2002) for **MTL** and extended to related logics in Esteva et al. (2002) (calling it Jenei and Montagna's method, briefly *JM method*), consists of showing that countable linearly ordered algebras of a given variety can be embedded into linearly and densely ordered members of the same variety, which can in turn be embedded into algebras with lattice reduct [0, 1]. This method works for t-norm (based) logics but seems to fail with associativity for **UL**. Thus, it does not appear to work in general for weakening-free *associative* logics based on uninorms such as **UL**. Because of this negative fact, Metcalfe and Montagna (2007) introduced a new approach for proving standard completeness of uninorm logics.[1)]

One interesting point to state is that JM method still works for non-associative logics (see Yang (2015)). Then, since we can introduce at least three weak forms of associativity (see Section

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1) This approach consists of the following two steps: 1. after extending logics with density rule, showing that such systems are complete w.r.t. linearly and densely ordered algebras, and for particular extensions are complete w.r.t. those algebras with lattice reduct [0, 1]; 2. giving a syntactic elimination of density rule (as a rule of the corresponding hypersequent calculus), i.e., showing that if ϕ is derivable in a uninorm logic **L** extended with density rule, then it is also derivable in **L**.

2), it raises the question as to whether the method also works for weakening-free weakly associative fuzzy logics or not, i.e.,

Does JM method work for (all, some, or none of) weakening-free micanorm logics with weak forms of associativity?

The answer is that it works for logics with the weak **t**-associativity, the *weakest* form of weak associativity introduced in Section 2, but does not for logics with other stronger forms of weak associativity introduced in Section 2. We shall verify this by providing standard completeness using construction in the style of Jenei-Montagna.

To gain our end, in Section 2 we introduce the wta-monoidal uninorm logic $\mathbf{WA_tMUL}$, which is intended to cope with the tautologies of left-continuous conjunctive wta-uninorms and their residua (introduced in Section 4), and its axiomatic extensions obtained by adding other stronger forms of weak associativity as *weakening-free weakly associative fuzzy logics*. In Section 3, we then define the algebraic structures corresponding to the systems, and provide algebraic completeness results for them. After introducing micanorms satisfying the corresponding weak forms of associativity in Section 4, in Section 5 we establish standard completeness for the system $\mathbf{WA_tMUL}$ and the other systems but with an additional axiom using JM method.

The systems, which will be investigated here, are all weakening-free. Thus, for simplicity, we henceforth call weakening-free fuzzy logics just *fuzzy logics*, if the context is clear. Also, for convenience, we shall adopt the notations and terminology similar to those in Cintula (2006), Esteva et al. (2002), and Yang (2009; 2015), and assume familiarity with them (together with results found therein).

2. Syntax

We base weakly associative fuzzy logics on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives →, &, ∧, ∨, and constants **T**, **F**, **f**, **t**. Further definable connectives are:

df1. $\neg\phi := \phi \rightarrow \mathbf{f}$,
df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define $\phi^{n}_{\mathbf{t}}$ as $\phi_{\mathbf{t}}$ & ⋯ & $\phi_{\mathbf{t}}$, n factors, where $\phi_{\mathbf{t}} := \phi \wedge \mathbf{t}$. For the rest of this paper, we use the customary notations and terminology, and the axiom systems to provide a consequence relation.

We start with the following axiom schemes and rules for the weak **t**-associative monoidal uninorm logic **WA$_t$MUL**, the basic weakly associative fuzzy logic defined here.

Definition 2.1 WA$_t$MUL consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi)$ ($\vee$-elimination, $\vee$-E)

A6. $\mathbf{F} \to \phi$ (ex falsum quodlibet, EF)

A7. $(\phi \& \psi) \to (\psi \& \phi)$ (&-commutativity, &-C)

A8. $\phi \leftrightarrow (\mathbf{t} \to \phi)$ (push and pop, PP)

A9. $\phi \to (\psi \to (\psi \& \phi))$ (&-adjunction, &-Adj)

A10. $(\phi_{\mathbf{t}} \& \psi_{\mathbf{t}}) \to (\phi \wedge \psi)$ ($\& \wedge$)

A11. $(\psi \& (\phi \& (\phi \to (\psi \to \chi)))) \to \chi$ (residuation, Res')

A12. $(\phi \to ((\phi \& (\phi \to \psi)) \& (\psi \to \chi))) \to (\phi \to \chi)$ (T')

A13. $((\delta \& \varepsilon) \to (\delta \& (\varepsilon \& (\phi \to \psi)_{\mathbf{t}}))) \vee (\delta' \to (\varepsilon' \to ((\varepsilon' \& \delta') \& (\psi \to \phi)_{\mathbf{t}})))$ (PL)

A14. $(\phi_{\mathbf{t}} \& (\psi_{\mathbf{t}} \& \chi_{\mathbf{t}})) \leftrightarrow ((\phi_{\mathbf{t}} \& \psi_{\mathbf{t}}) \& \chi_{\mathbf{t}})$ (weak **t**-associativity, $\mathrm{wAS}_{\mathbf{t}}$)

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi \vdash \phi_{\mathbf{t}}$ ($\mathrm{adj}_{\mathrm{u}}$)

$\phi \vdash (\delta \& \varepsilon) \to (\delta \& (\varepsilon \& \phi))$ (α)

$\phi \vdash \delta \to (\varepsilon \to ((\varepsilon \& \delta) \& \phi))$ (β).

Definition 2.2 A logic is an axiomatic extension (extension for short) of an arbitrary logic **L** if and only if (iff) it results from **L** by adding axiom schemes. In particular, the following are weakly associative fuzzy logics that extend $\mathbf{WA_tMUL}$:

- **t**-associative (ta-) monoidal uninorm logic $\mathbf{A_tMUL}$ is $\mathbf{WA_tMUL}$ plus

($\mathrm{AS}_{\mathbf{t}}$) $(\phi \& (\psi \& \chi))_{\mathbf{t}} \leftrightarrow ((\phi \& \psi) \& \chi)_{\mathbf{t}}$;

($\mathrm{RE}_{\mathbf{t}}$) $(\phi \to (\psi \to \chi))_{\mathbf{t}} \leftrightarrow ((\phi \& \psi) \to \chi)_{\mathbf{t}}$;

(SF_t) $(\phi \rightarrow \psi)_t \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$;
(PF_t) $(\psi \rightarrow \chi)_t \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$; and
(MT_t) $(\phi \rightarrow \psi)_t \rightarrow ((\phi \,\&\, \chi) \rightarrow (\psi \,\&\, \chi))$.

- (Yang (2009) Strong ta-monoidal uninorm logic **SA$_t$MUL** is **A$_t$MUL** plus
(sAS_t) $(\phi_t \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi_t \,\&\, \psi) \,\&\, \chi)$.[2)]

An easy computation shows the following.

Proposition 2.3 WA$_t$MUL proves:
(1) $\phi \rightarrow \psi \vdash (\phi \,\&\, \chi) \rightarrow (\psi \,\&\, \chi)$ (monotonicity, mt)
(2) $\phi \rightarrow (\psi \rightarrow \chi) \vdash \psi \rightarrow (\phi \rightarrow \chi)$ (permutation, pm)
(3) $\phi \rightarrow \psi \vdash \neg\psi \rightarrow \neg\phi$ (contraposition, cp)
(4) $\phi \rightarrow \neg\neg\phi$ (double negation introduction, DNI)
(5) $\neg(\phi \vee \psi) \leftrightarrow (\neg\phi \wedge \neg\psi)$ (de MorganI, DM1)
(6) $(\neg\phi \vee \neg\psi) \leftrightarrow \neg(\phi \wedge \psi)$ (de MorganII, DM2)
(7) $(\phi\wedge(\psi\vee\chi)) \leftrightarrow ((\phi\wedge\psi) \vee (\phi\wedge\chi))$ (distributivity, $\wedge\vee$-D)
(8) $(\phi\&(\psi\vee\chi)) \leftrightarrow ((\phi\&\psi)\vee(\phi\&\chi))$ ($\&\vee$-distributivity, $\&\vee$-D)
(9) $(\phi\&(\psi\wedge\chi)) \leftrightarrow ((\phi\&\psi)\wedge(\phi\&\chi))$ ($\&\wedge$-distributivity, $\&\wedge$-D)
(10) $(\phi_t \,\&\, \phi_t) \rightarrow \phi_t$ (**t**-square decreasing, SDE_t)
(11) $(\phi_t \rightarrow (\psi_t \rightarrow \chi))_t \leftrightarrow ((\phi_t \,\&\, \psi_t) \rightarrow \chi)_t$ (wRE_t)
(12) $(\phi \rightarrow \psi)_t \rightarrow ((\psi \rightarrow \chi)_t \rightarrow (\phi_t \rightarrow \chi_t))$ (wSF_t)
(13) $(\psi \rightarrow \chi)_t \rightarrow ((\phi \rightarrow \psi)_t \rightarrow (\phi_t \rightarrow \chi_t))$ (wPF_t)
(14) $(\phi \rightarrow \psi)_t \rightarrow ((\phi_t \,\&\, \chi_t) \rightarrow (\psi_t \,\&\, \chi_t))$ (wMT_t)

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2) Here the "**t**-associative" means that this logic satisfies the condition (d') for ta-uninorm in Definition 4.2 (iii) below and the "**A$_t$**" in each naming denote this fact. The "**W**" and "**S**" in **WA$_t$MUL**, and **SA$_t$MUL** mean "weaker than **t**-associativity" and "stronger than **t**-associativity", respectively.

(15) $(\phi \mathbin{\&} \mathbf{t}) \leftrightarrow \phi$
(16) $(\phi \to \psi) \vee (\psi \to \phi)$ (PL2)
(17) $(\phi \to \psi)_{\mathbf{t}} \vee (\psi \to \phi)_{\mathbf{t}}$ ($PL_{\mathbf{t}}$)
(18) $(\phi \wedge \mathbf{t}) \to (\mathbf{t} \vee \psi)$
(19) $((\phi \to \mathbf{F}) \mathbin{\&} \phi) \to \psi$
(20) $(\phi\wedge(\phi\to\mathbf{f})) \to (\psi\vee(\psi\to\mathbf{f}))$, i.e., $(\phi \wedge \neg\phi) \to (\psi \vee \neg\psi)$.

For easy reference we let Ls be a set of weakly associative fuzzy logics defined previously.

Definition 2.4 Ls = {$\mathbf{WA_tMUL}$, $\mathbf{A_tMUL}$, $\mathbf{SA_tMUL}$}.

In L ($\in$ Ls), **f** can be defined as $\neg\mathbf{t}$ and vice versa.

A *theory* over L ($\in$ Ls) is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using a rule of L. $T \vdash \phi$, more exactly $T \vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. A theory T is *inconsistent* if $T \vdash \mathbf{F}$; otherwise it is *consistent*.

The deduction theorem for L is as follows:

Proposition 2.5 Let T be a theory, and ϕ, ψ be formulas.
(i) (Cintula et al. (2013; 2015)) $T \cup \{\phi\} \vdash_L \psi$ iff $T \vdash_L \gamma(\phi) \to \psi$ for some $\gamma \in \Pi(bDT^*)$.[3]
(ii) (Yang (2009)) For L $\in$ {$\mathbf{A_tMUL}$, $\mathbf{SA_tMUL}$}, $T \cup \{\phi\} \vdash_L$

[3] For γ and $\Pi(bDT^*)$, see Cintula et al. (2013; 2015) and Yang (2015).

ψ iff there is n such that T $\vdash_L \phi^n_t \to \psi$.

For convenience, "$\neg$", "$\wedge$", "$\vee$", and "$\to$" are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

Remark 2.5 **MICAL** and **UL** are the systems as follows:

- (Yang (2015)) **MICAL** is **WA$_t$MUL** minus A14.
- (Metcalfe & Montagna (2007)) **UL** is **MICAL** plus

(PF) $(\psi \to \chi) \to ((\phi \to \psi) \to (\phi \to \chi))$; and

(RE) $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$.

Note that **UL** proves (AS) $(\phi \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi \,\&\, \psi) \,\&\, \chi)$. Thus, **WA$_t$MUL**, **A$_t$MUL**, and **SA$_t$MUL** can be regarded as *weakly associative generalizations* of **UL**.

3. Semantics

Suitable algebraic structures for L ($\in$ Ls) are obtained as varieties of residuated lattice-ordered unital groupoids (briefly, rlu-groupoids) in the sense of Galatos et al. (2007).

Definition 3.1 (Yang (2015)) (i) A *pointed bounded commutative rlu-groupoid* is a structure **A** = (A, $\top$, $\bot$, t, f, $\wedge$, $\vee$, $*$, $\to$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded lattice with top element $\top$ and bottom element $\bot$.

(Ⅱ) (A, *, t) is a commutative groupoid with unit.
(Ⅲ) $y \leq x \to z$ iff $x * y \leq z$, for all $x, y, z \in A$ (residuation).

(ii) An *MICAL-algebra* is a pointed bounded commutative rlu-groupoid satisfying: for all $x, y, z, w, z', w' \in A$,
(PL^{A}) $t \leq ((z*w) \to (z*(w*(x \to y)_t))) \vee (z' \to (w' \to ((w'*z')*(y \to x)_t)))$.

Definition 3.2 (L-algebras) A *WA_tMUL-algebra* is an MICAL-algebra satisfying: (wAS_t^{A}) $x_t * (y_t * z_t) = (x_t * y_t) * z_t$, for all $x, y, z \in A$; an *A_tMUL-algebra* is an MICAL-algebra satisfying: for all $x, y, z \in A$, (AS_t^{A}) $(x * (y * z))_t = ((x * y) * z)_t$, ($RE_t^{A}$) $(x \to (y \to z))_t = ((x * y) \to z)_t$, ($SF_t^{A}$) $(x \to y)_t \leq ((y \to z) \to (x \to z))$, ($PF_t^{A}$) $(y \to z)_t \leq ((x \to y) \to (x \to z))$, and ($MT_t^{A}$) $(x \to y)_t \leq ((x * z) \to (y * z))$; an *$SA_tMUL$-algebra* is an A_tMUL-algebra satisfying: (sAS_t^{A}) $x_t * (y * z) = (x_t * y) * z$, for all $x, y, z \in A$. We call all these algebras *L-algebras*.

A commutative unital groupoid (A, *, t) satisfying (associativity) $x * (y * z) = (x * y) * z$ on [0, 1] is a *uninorm* and this is a *t-norm* in case $t = \top$.

By x^n, we denote $x * \cdots * x$, n factors. Using $\to$ and f we can define t as $f \to f$, and $\neg$ as in (df1).

For L ($\in$ Ls), L-algebra (defined in Definition 3.2) is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An $\mathcal{A}$-*evaluation* is a function v : FOR → $\mathcal{A}$ satisfying: $v(\phi \to \psi) = v(\phi) \to v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{T}) = \top$, $v(\mathbf{F}) = \bot$, $v(\mathbf{f}) = f$, (and hence $v(\neg\phi) = \neg v(\phi)$ and $v(\mathbf{t}) = t$).

Definition 3.4 (Cintula (2006)) Let $\mathcal{A}$ be an L-algebra, T be a theory, ϕ be a formula, and K be a class of L-algebras.
(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an $\mathcal{A}$-*tautology* (or $\mathcal{A}$-*valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.
(ii) (Model) An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-*model* of T if $v(\phi) \geq t$ for each $\phi \in T$. We denote the class of $\mathcal{A}$-models of T, by *Mod(T,* $\mathcal{A}$*)*.
(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_K \phi$, if $Mod(T, \mathcal{A}) = Mod(T \cup \{\phi\}, \mathcal{A})$ for each $\mathcal{A} \in K$.

Definition 3.5 (**L**-algebra, Cintula (2006)) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is an ***L****-algebra* iff, whenever ϕ is L-provable in T (i.e. $T \vdash_L \phi$, L an L logic), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. $T \vDash_{\{\mathcal{A}\}} \phi$), $\mathcal{A}$ a corresponding L-algebra). By $MOD^{(l)}(L)$, we denote the class of (linearly ordered) **L**-algebras. Finally, we write $T \vDash^{(l)}_L \phi$ in place of $T \vDash_{MOD^{(l)}(L)} \phi$.

Theorem 3.6 (Strong completeness) Let T be a theory, and ϕ be a formula. $T \vdash_L \phi$ iff $T \vDash_L \phi$ iff $T \vDash^{l}_L \phi$.

Proof: We obtain this theorem as a corollary of Theorem 3.1.8 in Cintula & Noguera (2011). □

4. Weakly associative uninorms and their residua

In this section we define standard L-algebras based on the real unit interval [0, 1] and weakly associative (wa-) uninorms. Using 1, 0, e, and ∂, we shall express $\top$, $\bot$, identity t, and any f, respectively, on [0, 1].

Definition 4.1 An L-algebra is *standard* iff its lattice reduct is [0, 1].

In standard L-algebras the operator * is a wta-uninorm defined here.

Definition 4.2 (i) (micanorm, Yang (2015)) An *micanorm* is a function $\bigcirc$: $[0, 1]^2 \rightarrow [0, 1]$ such that for some $e \in [0, 1]$ and for all x, y, z $\in$ [0, 1]:

(a) $x \bigcirc y = y \bigcirc x$ (commutativity),

(b) $e \bigcirc x = x$ (identity),

(c) $x \leq y$ implies $x \bigcirc z \leq y \bigcirc z$ (monotonicity).

(ii) (wta-uninorm) A *wta-uninorm* is an micanorm satisfying: for all x, y, z $\in$ [0, 1],

(d) $x,y,z \leq e$ implies $x\bigcirc(y\bigcirc z) = (x\bigcirc y)\bigcirc z$ (wt-associativity).

(iii) (ta-uninorm) A ta-uninorm is an micanorm satisfying: for all

$x, y, z \in [0, 1]$,

(d') $x \circ (y \circ z), (x \circ y) \circ z \leq e$ implies $x \circ (y \circ z) = (x \circ y) \circ z$ (t-associativity).

(iv) (sta-uninorm) An sta-uninorm is an micanorm satisfying: for all $x, y, z \in [0, 1]$,

(d'') $x \leq e$ implies $x \circ (y \circ z) = (x \circ y) \circ z$ (st-associativity).

By *weakly associative (wa-) uninorm(s)*, we ambiguously denote these wta-uninorms collectively if we need not distinguish them.

A wa-uninorm is called *conjunctive* if $0 \circ 1 = 0$, and disjunctive if $0 \circ 1 = 1$. Associative micanorm is a *uninorm*; and uninorm satisfying that $e = 1$ is a *t-norm*.

The left-continuity property of conjunctive wa-uninorms is important in the sense that it gives rise to a residuated implication and so plays an important role in standard completeness proof of Ls as in t-norm and uninorm based logics such as **MTL**, **UL**, etc., (cf. see Esteva & Godo (2001), Esteva et al. (2002), Jenei & Montagna (2002), and Metcalfe & Montagna (2007)). $\circ$ is said to be *residuated* if there is $\Rightarrow : [0, 1]^2 \rightarrow [0, 1]$ satisfying (residuation) on [0, 1]. Then, given a wa-uninorm $\circ$, *residuated implication* (briefly R-implication)} $\Rightarrow$ determined by $\circ$ is defined as $x \Rightarrow y := \sup\{z \in [0, 1]: x \circ z \leq y\}$ for all $x, y \in [0, 1]$. Then, as in uninorms, we can show that for any wa-uninorm $\circ$, $\circ$ and its R-implication $\Rightarrow$ form a residuated pair iff $\circ$ is conjunctive and left-continuous in both arguments (cf. see Proposition 5.4.2 in Gottbald (2001)).

It is clear that the operation * of any WA_tMUL-algebra on [0, 1] is a conjunctive wta-uninorm with identity t and residuum ⇒; conversely any residuated wta-uninorm gives rise to a WA_tMUL-algebra on [0, 1] as follows:

Proposition 4.3 If ○ is a wta-uninorm with residuum ⇒ and identity e, then for any $\partial \in [0, 1]$, ([0, 1], 1, 0, e, ∂, min, max, ○, ⇒) is a WA_tMUL-algebra on [0, 1].

Proof: It is clear that ([0, 1], 1, 0, e, ∂, *min*, *max*, ○, ⇒) is an SL-algebra. Furthermore, since ○ is a wta-uninorm, it satisfies (wt-associativity). We prove that it satisfies (PL^A). For x, y $\in$ [0, 1], if x $\leq$ y, then $e \leq x \Rightarrow y$ and so $e = (x \Rightarrow y)_e$. Furthermore, using (monotonicity) and (residuation), $e \leq w \Rightarrow (w \circ (x \Rightarrow y)_e)$, and so $e \leq (z \circ w) \Rightarrow (z \circ (w \circ (x \Rightarrow y)_e))$; therefore, $e \leq ((z \circ w) \Rightarrow (z \circ (w \circ (x \Rightarrow y)_e))) \vee (z' \Rightarrow (w' \Rightarrow ((w' \circ z') \circ (y \Rightarrow x)_e)))$. Let y $\leq$ x. Similarly, we can obtain $e \leq z' \Rightarrow (w' \Rightarrow ((w' \circ z') \circ (y \Rightarrow x)_e))$; therefore, $e \leq ((z \circ w) \Rightarrow (z \circ (w \circ (x \Rightarrow y)_e))) \vee (z' \Rightarrow (w' \Rightarrow ((w' \circ z') \circ (y \Rightarrow x)_e)))$. This ensures that this algebra satisfies (PL^A). □

Similarly we can show that: if ○ is a ta-uninorm (sta-uninorm resp) with residuum ⇒ and identity e, then for any $\partial \in$ [0, 1], ([0, 1], 1, 0, e, ∂, min, max, ○, ⇒) is an A_tMUL-algebra (SA_tMUL-algebra resp) on [0, 1].

5. Standard completeness

In this section we provide *standard* completeness results for Ls (extended with (R-C) $((\phi \,\&\, \psi) \rightarrow (\phi \wedge \psi)) \vee ((\phi \vee \psi) \rightarrow (\phi \,\&\, \psi))$ w.r.t. **A$_t$MUL** and **SA$_t$MUL**). We first show that finite or countable linearly ordered **WA$_t$MUL**-algebras are embeddable into a standard algebra. (For convenience, we add less than relation symbol to such algebras.)

Proposition 5.1 For every finite or countable linearly ordered **WA$_t$MUL**-algebra **A** = $(A, \leq_A, \top, \bot, t, f, \wedge, \vee, *, \Rightarrow)$, there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the following conditions hold:

(I) X is densely ordered, and has a maximum Max, a minimum Min, and special elements e, ∂.

(II) $(X, \circ, \preccurlyeq, e)$ is a linearly ordered monotonic commutative wta-groupoid with unit.

(III) $\circ$ is conjunctive and left-continuous w.r.t. the order topology on $(X, \preccurlyeq)$.

(IV) h is an embedding of the structure $(A, \leq_A, \top, \bot, t, f, \wedge, \vee, *)$ into $(X, \preccurlyeq, \mathrm{Max}, \mathrm{Min}, e, \partial, \min, \max, \circ)$, and for all m, n $\in$ A, $h(m \Rightarrow n)$ is the residuum of h(m) and h(n) in $(X, \preccurlyeq, \mathrm{Max}, \mathrm{Min}, e, \partial, \max, \min, \circ)$.

Proof: For convenience, we assume A as a subset of $Q \cap [0, 1]$ with finite or countable elements, where 0 and 1 are least and greatest elements and some e and any ∂ are special elements, each of which corresponds to $\top$, $\bot$, t, and f, respectively. Let

$$X = \{(m, x): m \in A \setminus \{0\ (= \bot)\} \text{ and } x \in \mathbf{Q} \cap (0, m]\} \cup \{(0, 0)\}; \text{ and}$$

for (m, x), (n, y) $\in$ X,

$(m, x) \leq (n, y)$ iff either $m <_A n$, or $m =_A n$ and $x \leq y$.

For convenience, we henceforth drop the index A in $\leq_A$ and $=_A$, if we need not distinguish them. But context should clarify what we mean.

Define for (m, x), (n, y) $\in$ X:

$$(m,x) \circ (n,y) = \begin{cases} \max\{(m,x), (n,y)\} & \text{if } m*n = m \vee n,\ m \neq n, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \\ \min\{(m,x), (n,y)\} & \text{if } m * n = m \wedge z, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \\ (m * n, m * n) & \text{otherwise.} \end{cases}$$

Here, we just prove that $\circ$ satisfies the wt-associativity of (II).

wt-Associativity: we assume that (l, x), (m, y), (n, z) $\leqslant e$ and show that $(l, x) \circ ((m, y) \circ (n, z)) = ((l, x) \circ (m, y)) \circ (n, z)$ (wAS_t^A). Without further mention, we will use the fact that * is weakly t-associative. We distinguish several cases:

● Case (i). $l * (m * n) = \vee(l, m, n)$. Since $l * (m * n) \leq$

e, l = m = n $\leq$ e. Then both sides of (wAS_t^A) are equal to min{(l, x), (m, y), (n, z)}.

● Case (ii). l * (m * n) = ∧(l, m, n). Since l * (m * n), (l * m) * n $\leq$ e, both sides of (wAS_t^A) are equal to min{(l, x), (m, y), (n, z)}.

● Case (iii). l * (m * n) ≠ ∨(l, m, n), ∧(l, m, n), and l * (m * n) ∈ {l, m, n}. Since l, m, n $\leq$ e and so l * (m * n) $\leq$ l ∧ m ∧ n, this is not the case.

● Case (iv). l * (m * n) ∉ {l, m, n} and either l * (m * n) = l ∨ (m * n) = m * n or l * (m * n) = l ∧ (m * n) = m * n. The first one is not the case as in Case (iii). Consider the second one. This implies that (l, x) ○ ((m, y) ○ (n, z)) = min{(l, x), (m, y) ○ (n, z)} = (m, y) ○ (n, z) = (m * n, m * n). Thus, both sides of (wAS_t^A) are equal to (m * n, m * n).

● Case (v). l * (m * n) ∉ {l, m, n} and l * (m * n) ≠ l ∨ (m * n), l ∧ (m * n). Then, we need to consider the case that l * (m * n) $\leq$ e. Then, since l * (m * n) < l ∧ m ∧ n, both sides of (wAS_t^A) are equal to (l * (m * n), l * (m * n)).

Proof of the remaining is almost same as Proposition 2 in Yang (2015). □

Proposition 5.2 Every countable linearly ordered

$\mathbf{WA_tMUL}$-algebra can be embedded into a standard algebra.

Proof: Its proof is almost the same as Proposition 3 in Yang (2015). □

Theorem 5.3 (Strong standard completeness) For $\mathbf{WA_tMUL}$, the following are equivalent:

(1) $T \vdash_{\mathbf{WAtMUL}} \phi$.

(2) For every standard $\mathbf{WA_tMUL}$-algebra and evaluation v, if $v(\psi) \geq e$ for all $\psi \in T$, then $v(\phi) \geq e$.

Proof: (1) to (2) follows from definition. We prove (2) to (1). Let ϕ be a formula such that $T \nvdash_{\mathbf{WAtMUL}} \phi$, $\mathbf{A}$ be a linearly ordered $\mathbf{WA_tMUL}$-algebra, and v be an evaluation in $\mathbf{A}$ such that $v(\psi) \geq t$ for all $\psi \in T$ and $v(\phi) < t$. Let h' be the embedding of $\mathbf{A}$ into the standard $\mathbf{WA_tMUL}$-algebra as in proof of Proposition 2 in Yang (2015). Then $h' \circ v$ is an evaluation into the standard $\mathbf{WA_tMUL}$-algebra such that $h' \circ v(\psi) \geq e$ and yet $h' \circ v(\phi) < e$. □

Theorem 5.3 ensures that $\mathbf{WA_tMUL}$ is complete w.r.t. left-continuous conjunctive wta-uninorms.

Remark 5.4 Note that the definitions of $\circ$ and X in the proof of Proposition 5.1 fail with (t-associativity) for $\mathbf{A_tMUL}$. For instance, let $l * (m * n) \neq \vee(l, m, n), \wedge(l, m, n)$, and $l * (m * n) \in \{l, m, n\}$. Consider the case that $l * (m * n) = l \vee (m$

$* n) = l = m * n$ and $m < l < e < n$. $(l, x) \circ ((m, y) \circ (n, z)) = \min\{(l, x), (m * n, m * n)\} = (l, x)$. But if $l * m = m$ and $(l, x) < (l, l)$, then $((l, x) \circ (m, y)) \circ (n, z) = (m * n, m * n)$ $(= (l, l)) \neq (l, x)$. This implies that this method, as far as it still has such definitions, fails not merely for **A_tMUL** but also for **SA_tMUL** and **UL** Therefore, JM method does not appear to work for these two weakly associative systems **A_tMUL** and **SA_tMUL**.

Notice that, while JM method does not appear to work for **UL**, it still works for **MTL**, i.e., **UL** plus (W). Similarly, this method may work for some particular extensions of **A_tMUL** and **SA_tMUL**. As examples, we consider **A_tMUL** and **SA_tMUL**, respectively, extended with (R-C) above.

Let ***A_tMULr*** (***SA_tMULr*** resp) be **A_tMUL** (**SA_tMUL** resp) plus (R-C). We define an A_tMULr-algebra (an SA_tMULr-algebra resp) to be an A_tMUL-algebra (an SA_tMUL-algebra resp) satisfying: (reinforcement without compensation, rc) for all $x, y \in A$,

$$t \leq ((x * y) \Rightarrow (x \wedge y)) \vee ((x \vee y) \Rightarrow (x * y)); \text{ and}$$

an rc-wa-uninorm to be a wa-uninorm satisfying (rc) on [0, 1]. Notice that in linearly ordered algebras using (rc) we obtain that: (rc') for all $x, y \in A$,

$$x * y \leq \min\{x, y\} \text{ or } \max\{x, y\} \leq x * y.$$

Then, in an analogy to the above, we can show the following.

Theorem 5.5 (Strong standard completeness) For L $\in$ {**A$_t$MULr**, **SA$_t$MULr**}, the following are equivalent:

(1) T $\vdash_L$ ϕ.

(2) For every standard **L**-algebra and evaluation v, if v(ψ) $\geq$ e for all $\psi \in$ T, then v(ϕ) $\geq$ e.

Proof: First define: for (m, x), (n, y) $\in$ X,

$$(m,x) \circ (n,y) = \begin{cases} \max\{(m,x), (n,y)\} & \text{if } m * n =_A m \vee n \text{ and } (m, x) \succ e \text{ or } (n, y) \succ e; \\ \min\{(m,x), (n,y)\} & \text{if } m * n = m \wedge z, \text{ and } (m, x) \preceq e \text{ or } (n, y) \preceq e; \\ (m * n, m * n) & \text{otherwise.} \end{cases}$$

For **A$_t$MULr** (**SA$_t$MULr** resp), we prove (rc) of $\circ$ and t-associativity (st-associativity resp) as follows:

$\circ$-rc: we instead prove that for all (m, x), (n, y) $\in$ X, (m, x) $\circ$ (n, y) $\preccurlyeq$ min{(m, x), (n, y)} or max{(m, x), (n, y)} $\preccurlyeq$ (m, x) $\circ$ (n, y), i.e., (rc') of $\circ$ ($\circ$-rc'). Note that by (rc') of *, m * n $\leq$ m $\wedge$ n or m $\vee$ n $\leq$ m * n. If m * n = m $\vee$ n, and (m, x) $\succ$ e or (n, y) $\succ$ e, then (m, x) $\circ$ (n, y) = max{(m, x), (n, y)}. If m * n = m $\wedge$ n, and (m, x) $\preccurlyeq$ e or (n, y) $\preccurlyeq$ e, then (m, x) $\circ$ (n, y) = min{(m, x), (n, y)}. Otherwise, by (rc') of *, (m, x) $\circ$ (n, y) = (m * n, m * n) $\prec$ min{(m, x), (n, y)} or $(m, x) \circ (n, y) = (m * n, m * n) \succ \max\{(m, x), (n, y)\}$.

t-Associativity for **A$_t$MULr**: We assume that $(l, x) \circ ((m, y) \circ (n, z)), ((l, x) \circ (m, y)) \circ (n, z) \preccurlyeq e$. We need to show that $(l, x) \circ ((m, y) \circ (n, z)) = ((l, x) \circ (m, y)) \circ (n, z)$ (AS_t^A). Its proof is analogous to that of (wt-associativity) in Proposition 5.1.

st-Associativity for **SA$_t$MULr**: We assume that $(l, x) \preccurlyeq e$ and show that $(l, x) \circ ((m, y) \circ (n, z)) = ((l, x) \circ (m, y)) \circ (n, z)$ (sAS_t^A). Without further mention, we will use the fact that * is strong t-associative. We distinguish several cases:

● Case (i). $l * (m * n) = \vee(l, m, n)$.

◇ Subcase (i-a). $m * n = m \vee n$.

(a-1) $(l, x) \succ e$ or $(m, y) \circ (n, z) \succ e$. Since $(l, x) \preccurlyeq e$, $(m, y) \circ (n, z) \succ e$ and so $(m, y) \succ e$ or $(n, z) \succ e$. Then, both sides of (sAS_t^A) are equal to $\max\{(l, x), (m, y), (n, z)\}$.

(a-2) $(l, x), (m, y) \circ (n, z) \preccurlyeq e$. This implies $l = m = n \leq e$. Thus both sides of (sAS_t^A) are equal to $\min\{(l, x), (m, y), (n, z)\}$.

◇ Subcase (i-b). $m * n = m \wedge n$.

(b-1) $(l, x) \preccurlyeq e$ or $(m, y) \circ (n, z) \preccurlyeq e$. If $e < m = n$, then $(m, y) \circ (n, z) = \max\{(m, y), (n, z)\}$ and so both sides of (sAS_t^A) are equal to $\max\{(l, x), (m, y), (n, z)\}$. Otherwise, i.e., if

$m \leq e$ or $n \leq e$, then $(m, y) \circ (n, z) = \min\{(m, y), (n, z)\}$ and so both sides of (sAS_t^A) are equal to $\min\{(l, x), (m, y), (n, z)\}$.

(b-2) $(l, x), (m, y) \circ (n, z) > e$. This is not the case by the supposition.

◇ Subcase (i-c). $m * n \neq m \vee n, m \wedge n$.

This is not the case because $(l, x) \leqslant e$ and so $(m, y) \circ (n, z) \leqslant e$.

● Case (ii). $l * (m * n) = \wedge(l, m, n)$. Its proof is analogous to that of Case (i).

● Case (iii). $l * (m * n) \neq \vee(l, m, n), \wedge(l, m, n)$, and $l * (m * n) \in \{l, m, n\}$. This is not the case because $\vee(l, m, n) \leq l * (m * n)$ or $l * (m * n) \leq \wedge(l, m, n)$ by (rc) of *.

● Case (iv). $l * (m * n) \notin \{l, m, n\}$ and either $l * (m * n) = l \vee (m * n) = m * n$ or $l * (m * n) = l \wedge (m * n) = m * n$. Let the first be the case. Then both sides of (sAS_t^A) are equal to $(m * n, m * n)$ since $l, m \vee n < m * n$ by (rc) of *. Let the second be the case. Then since $l \wedge (m * n) = m * n$ and $(l, x) \circ ((m, y) \circ (n, z)) = \min\{(l, x), (m, y) \circ (n, z)\} = (m, y) \circ (n, z)$, both sides of (sAS_t^A) are equal to $(m * n, m * n)$.

● Case (v). $l * (m * n) \notin \{l, m, n\}$ and $l * (m * n) \neq l \vee (m * n), l \wedge (m * n)$. We need to consider the cases $l * (m * n) > e$ and $l * (m * n) \leq e$. But the first is not the case since $l \leq e$ and so $l * (m * n) \leq l \vee (m * n)$. If the second is the case, since $l * (m * n) < \wedge(l, m, n)$, both sides of ($sAS_t^A$) are equal to $(l * (m * n), l * (m * n))$.

Proof of the remaining is analogous to that of **WA$_t$MUL**. □

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전북대학교 철학과, 비판적사고와논술연구소

Department of Philosophy & Institute of Critical Thinking and Writing, Chonbuk National University

eunsyang@jbnu.ac.kr

약한 결합 원리를 갖는 퍼지 논리

양 은 석

이 글에서 우리는 (곱 연언 &의) 약한 형식의 결합 원리를 갖는 약화 없는 퍼지 논리를 연구한다. 이를 위하여 먼저 wta-유니놈에 기반 한 체계 $\mathbf{WA_tMUL}$과 이의 두 공리적 확장 체계들을 약화 없는 약한 결합 원리를 갖는 퍼지 논리로 소개한다. 그리고 각 체계에 상응하는 대수적 구조를 정의한 후, 이 체계들이 대수적으로 완전하다는 것을 보인다. 다음으로 제네이-몬테그나 스타일의 구성 방식을 사용하여 체계 $\mathbf{WA_tMUL}$과 추가적 공리를 갖는 두 확장 체계들이 표준적으로 완전하다는 것을 보인다.

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