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On Induction Principles in Frege's *Grundgesetze* and in Systems Thereafter*

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【Abstract】We compare the approaches to natural numbers and the induction principles in Frege's *Grundgesetze* and in systems thereafter. We start with an illustration of Frege's approach and then explain the use of induction principles in Zermelo-Fraenkel set theory and in modern type theories such as Calculus of Inductive Constructions. A comparison among the different approaches to induction principles is also given by analyzing them in respect of predicativity and impredicativity.

【Key Words】Induction principles, Frege, Grundgesetze, set theory, type theory.

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1. Introduction

During the 19th century, there was an increasing tendency towards exacter investigation in the foundation of mathematics. Around the turn of the 20th century, logicians started to realize that ordinary mathematical arguments can be represented in formal axiomatic systems. One of the main figures in this tendency was Gottlob Frege. His concern was twofold: whether arithmetical judgments can be proved in a purely logical manner and how far one could go in arithmetic by merely using the laws of logic.

Beginning with *Begriffsschrift* (Frege 1879), Frege's main concern was to reduce everything used in arithmetic to pure logic, and for that purpose he invented a special kind of language[1)] where statements (of arithmetic) can be proved as true based only upon general logical laws and definitions. In the two volumes of *Grundgesetze der Arithmetik* (1893, 1903) he showed for instance how to define natural numbers and proved that the basic axioms of arithmetic can be derived.

There is a point by which one can distinguish Frege's approach to natural numbers from those of other logicians. It is the way of dealing with *the principle of induction* (*IND*). In *Fregean Arithmetic*, *IND* is an immediate consequence of the definition of natural numbers while e.g. for Dedekind, Peano, and Hilbert it is

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1) "I wanted to supplement the formula-language of mathematics with signs for logical relations so as to create a concept-script which would make it possible to dispense with words in the course of proof, and thus ensure the highest degree of rigour whilst at the same time making the proofs as brief as possible." Cf. [Frege 1969, p.53].

postulated as an axiom. One could say Frege accepted IND as a general principle when doing arithmetic. For more about Frege's understanding of numbers we refer to [박준용 2007].

In this paper, we compare the approaches to natural numbers and the induction principles in Frege's Grundgesetze and in systems thereafter. We start with an illustration of Frege's approach and then explain the use of induction principles in Zermelo-Fraenkel set theory and in modern type theories such as Calculus of Inductive Constructions. A comparison among the different approaches to induction principles is also given by analyzing them in respect of predicativity and impredicativity.

2. Induction principle for Frege

Frege creates in *Begriffsschrift* a special language where statements (of arithmetic) can be proved to be true based only upon general logical laws and definitions.[2] The *Begriffsschrift* can be used as a basis for forming characteristic languages which have intuitive content. Its extension for arithmetic, for example, happens by adding arithmetic concepts and axioms as further principles.

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2) "… in a *lingua characteristica* the relationship between an expression and its content is supposed to be arbitrary at most at level of primitive expressions and concepts. Expressions of complex concepts should be built up from simple ones in a systematic manner so that to know the content of any expression of the *lingua characteritica*, one needs to know only fundamental terms and the method of their combination. The *Begriffsschrift* is such a language." (Korte 2010, p.292)

In the first volume of *Grundgesetze*, Frege presented such an extension where arithmetic concepts can be defined and truths of propositions can be formally proved. Unfortunately, the formal system of *Grundgesetze* is inconsistent. It is because Russell's Paradox can be derived from the infamous *Axiom V*:

$$(\hat{\alpha} f(\alpha) = \hat{\varepsilon} g(\varepsilon)) \leftrightarrow \forall x\, (f(x) = g(x))$$

Here $\hat{\alpha} f(\alpha)$ denotes the value-range[3] of the function f. This axiom asserts that the value-range of the function f is identical to the value-range of the function g if and only if f and g map every object to the same value.

Although the inconsistency is widely known, the *Grundgesetze* contains all the essential steps necessary to prove the fundamental propositions of arithmetic from a single, *nearly consistent* principle. This principle, known as *Hume's Principle*, asserts that for any concepts F and G, the number of objects satisfying F is equal to that of objects satisfying G if and only if there is an one-to-one correspondence between the objects satisfying F and the objects satisfying G. *Hume's Principle* is nearly consistent in the sense that *Fregean Arithmetic*[4] is equi-consistent with second-order arithmetic.

[Heck 1995], for example, points out that Frege's essential use of *Axiom V* is made only in the proof of *Hume's Principle* and

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3) Value-range of a function can be thought of its graph in the set-theoretic sense.

4) *Fregean Arithmetic* is the second-order theory whose sole non-logical axioms is *Hume's Principle*. For a proof of the equi-consistency, see [Boolos 1987].

that *Frege's Theorem* holds, i.e. *the five Dedekind-Peano axioms* for number theory can be derived from *Hume's Principle* in second-order logic. The five Dedekind-Peano axioms are the following:

- Zero is a number.
- Zero isn't the successor of any number.
- No two numbers have the same successor.
- The principle of mathematical induction holds.
- Every number has a successor.

Among the five axioms, however, the principle of mathematical induction is a trivial consequence of the definition of natural numbers. To understand why, we need to know two concepts: the *predecession* relation and the *ancestral* of a relation.

2.1. Natural numbers and induction

To define natural numbers, Frege starts with the definition of predecession relation. x *(immediately) precedes* y when there is a concept F and an object w such that: (a) w satisfies F, (b) y is the number of objects satisfying F and (c) x is the number of objects other than w satisfying F. In formal terms:

$$\begin{aligned} PRED(x,y) := \quad & \exists F \exists w [Fw \wedge \\ & \qquad y = Nz: Fz \wedge \\ & \qquad x = Nz: (Fz \wedge z \neq w)] \end{aligned}$$

Here $Nz : Fz$ denotes the number (cardinality) of objects satisfying F.

Frege makes use also of the definition of the *(strong) ancestral* of a relation. Given a relation R, *x comes before y in the R-series* when y satisfies all those R-hereditary concepts F which is satisfied by every object to which x is R-related. In formal terms:

$$R^{+}(x,y) := \forall F[\forall z\,(R(x,z) \rightarrow Fz) \wedge Her(F,R) \rightarrow Fy]$$

where

$$Her(F,R) := \forall x\, y\,(R(x,y) \wedge Fx \rightarrow Fy)\ .$$

The *weak ancestral of a relation* is defined as follows: *y is a member of the R-series beginning with x* when either x comes before y in the R-series or $x = y$. In formal terms:

$$R^{*}(x,y) := R^{+}(x,y) \vee x = y$$

The two concepts, predecession and weak ancestral play a very important role for Frege in defining natural numbers. First, he defined the number 0 as the number of objects which are not self-identical:

$$0 := Nx : x \neq x$$

The concept of a natural number is then now defined as:

$$\mathbb{N}\ x := \ PRED^{*}(0,\, x)\ .$$

So an object x is a natural number when it is a member of the $PRED$-series beginning with 0. Note that the principle of mathematical induction follows immediately as a tautology:

$$IND := \ \forall F[F0 \rightarrow \ Her(F, PRED) \rightarrow \ \forall x\,(\mathbb{N}\ x \rightarrow \ Fx)]$$

Note also that we may define the first transfinite number as

$$\infty := \ Nx : \ \mathbb{N}\ x$$

called “Endloss” (“endless” in English) in Grundgesetze (Vol. I, p. 150). For a detailed discussion about Fregean Arithmetic, we refer to [이종권 2000].

2.2. Definition by induction

[Heck 1995] notices also that Theorem 256 in Grundgesetze (Vol. I, p. 197) is actually a version of what is known as the recursion theorem up to ω[5] which is usually stated in modern notation as follows:

Given a function $g : A \rightarrow A$ and an object $a \in A$, there is a unique function $\varphi : \mathbb{N} \rightarrow A$ such that $\varphi(0) = a$ and $\varphi(Suc\, n) = g(\varphi(n))$.

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[5] ω denotes the first infinite, countable ordinal.

where $Suc: \mathbb{N} \rightarrow \mathbb{N}$ is the successor function whose existence is guaranteed by the validity of *Frege's theorem*.

One of the most famous examples is the definition of addition where $g = Suc$:

$$a + 0 = a$$
$$a + (Suc\, n) = Suc\,(a + n)$$

3. Natural numbers and induction in ZF set theory

Zermelo-Fraenkel (ZF) set theory consists of the following axioms:

(*Comp*)	$\forall z \exists y \forall x\, [\, x \in y \leftrightarrow (x \in z \vee \varphi(x)\,]$
(*Found*)	$\forall x\, [\exists y\, (y \in x) \rightarrow \exists y\, (y \in x \wedge \neg \exists z\, (z \in y \wedge z \in x))]$
(*Ext*)	$\forall x\, y\, [\, \forall z\, (z \in x \leftrightarrow z \in y\,) \rightarrow x = y\,]$
(*Empty*)	$\exists x \forall y\, (\neg y \in x)$
(*Infty*)	$\exists x [\varnothing \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x)]$
(*Pair*)	$\forall x\, y \exists z\, [\, x \in z \wedge y \in z\,]$
(*Union*)	$\forall x \exists y \forall z\, [\, z \in y \leftrightarrow \exists v \in x\, (v \in x \wedge z \in v)]$
(*Power*)	$\forall x \exists y \forall z\, [\, z \in y \leftrightarrow z \subseteq x\,]$
(*Coll*)	$\forall x\, [\, \forall y \in x \exists ! z\, \varphi(y, z) \rightarrow \exists v \forall y \in x \exists z \in v\, \varphi(y, z)]$

Here $\{y\}$ denotes the singleton containing as elements only the set y, and $\varnothing$ denotes the empty set whose existence is guaranteed by the axiom (*Empty*).

It is well known that ZF is a system sufficient for guaranteeing the existence of natural numbers, the set of all natural numbers, and definition by induction. Here we summarize which axioms play the main role.

3.1. Natural numbers

In ZF set theory, the axiom (*Infty*) mimics Kronecker's understanding of natural numbers: *God made the integers, all else is the work of man.*[6] Indeed, the natural numbers are usually defined by

$$0 := \varnothing \quad \text{and} \quad Suc\, n := n \cup \{n\}\,,$$

and the set $\mathbb{N}$ of all natural numbers can be defined as the smallest set satisfying (*Infty*) whose existence can be assured by the comprehension axiom scheme (*Comp*).

One should notice however that the definition of natural numbers implicitly assume the existence of the concept of natural numbers.

3.2. Transfinite induction and recursion

There is no axiom explicitly mentioning induction. However, it is well known that in ZF, the principle of transfinite induction (TI) along ordinals is allowed:

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[6] [Weber 1893] writes in page 15 that Kronecker made this legendary remark during a lecture to the Berliner Naturforscher-Versammlung in 1886.

(TI) $\varphi(0) \to$
$\forall\,\alpha(\varphi(\alpha) \to \varphi(Suc\,\alpha)) \to$
$\forall\,\lambda{\in}\mathrm{Lim}[(\forall\,\xi{\in}\lambda\,\varphi(\xi)) \to \varphi(\lambda)] \to$
$\forall\,\alpha \in Ord\,\varphi(\alpha)$

Here Lim is the class of limit ordinals and Ord is the class of all ordinals.

Moreover, the principle of transfinite recursion (TR) is also available: Let V be the class of all sets. Given a set $a \in V$, and two class functions $G, H: V \to V$, there is a unique function $F: Ord \to V$ such that

$$\text{(TR)} \qquad \begin{array}{rcl} F(0) & = & a \\ F(Suc\,\alpha) & = & G(F(\alpha)) \\ F(\lambda) & = & H(F \restriction \lambda), \end{array}$$

where λ stands for an infinite ordinals and $F \restriction \lambda$ denotes the function $\{(\alpha, F(\alpha)) \mid \alpha < \lambda\}$ which is the restriction of F to the domain λ.

4. Inductive definitions in logic and mathematics

In everyday exercise of logic and mathematics we often encounters definitions by induction. For example, the terms in the language of first-order logic is defined inductively. Given a first-order language L, the set of terms is defined as follows:

- *Variables and constants are terms.*
- *If* $t_1, ..., t_n$ *are terms and* f *is an* n*-ary function symbol, then* $f(t_1, ..., t_n)$ *is a term.*

Then we talk about the set of all terms in the language L.

This kind of construction can be found everywhere in logic and mathematics. Below are some examples from logic and mathematics.

- The inductive definition of formulas of the language L.
- The inductive definition of derivation as a predicate for provable formulas.
- The smallest subgroup containing a subset of a group.
- The construction of a basis for a vector space.

It remains, however, the question of the existence of sets containing all and only the objects described by the inductive definitions. Does the ZF set theory provide the foundation for it? How are the principle of transfinte induction and the principle of transfinite recursion related to this question? In the rest of this paper, we will introduce two foundational approaches to these questions: set-theoretic one and type-theoretic one.

4.1. Set-theoretic approach: Aczel's rule sets

Note that definition by induction typically has the following form

$$(*) \qquad \frac{J_1 \ \cdots \ J_k}{J}$$

where $k \geqq 0$ and $J_1 \cdots J_k$ are *premisses* and J is the *conclusion* of the instance of a rule. In case $k=0$, there are no premisses.

Let's, for example, have a careful look at the definition of terms in a first-order language L:

a) Variables and constants are terms.
b) If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term.

For the rule a) we have $k=0$ and J says that variables and constants are terms. In case of b), we have

- $k=n+1$,
- for each $i \leqq n$, J_i says that t_i is a term,
- J_{n+1} says that f is an n-ary function, and
- J says that $f(t_1, \ldots, t_n)$ is a term.

[Aczel 1977] notices that the form **(*)** corresponds exactly to his concept of *rules*. A *rule* is a pair (X, y) where X is a set, called the set of premisses and y is the conclusion. The rule (X, y) will usually be written $X \rightarrow y$. A *rule set* is a set of rules. Given a rule set Ψ, a set A is called *Ψ-closed* if, for any $X \rightarrow y \in \Psi$, $y \in A$ follows from $X \subseteq A$, that is, if premisses are in A so is the conclusion. Note that there is the least Ψ -closed set

$$I_{\Psi} := \bigcap \{A \mid A \text{ is } \Psi\text{-closed}\}.$$

In fact, each rule set Ψ generates a monotone operator

$$\Gamma_{\Psi}(A) := \{y \mid \text{there exists some } X \subseteq A \text{ such that } X \to y \in \Psi\}$$

such that I_{Ψ} is the least fix-point of Γ_{Ψ}, i.e. $\Gamma_{\Psi}(I_{\Psi}) = I_{\Psi}$. Furthermore, it is well known that using (TR) one can construct I_{Ψ} as follows

$$\begin{array}{rcl} I_{\Psi}^{0} & := & \Gamma_{\Psi}(\varnothing) \\ I_{\Psi}^{\alpha+1} & := & \Gamma_{\Psi}(I_{\Psi}^{\alpha}) \\ I_{\Psi}^{\lambda} & := & \Gamma_{\Psi}(\bigcup_{\alpha<\lambda}(I_{\Psi}^{\alpha}) \end{array}$$

and $I_{\Psi} = I_{\Psi}^{|\Gamma_{\Psi}|}$ where $|\Gamma_{\Psi}|$ denote the least ordinal such that $I_{\Psi}^{\alpha} = \bigcup_{\xi<\alpha} I_{\Psi}^{\xi}$.

Example 1. The set $\mathbb{N}$ of natural numbers is I_{Ψ_N}, where

$$\Psi_N = \{\varnothing \to 0\} \cup \{\{n\} \to n \cup \{n\} \mid n \text{ arbitrary set}\}$$

Here, 0 stands for $\varnothing$.

Example 2. Given a group G and an arbitrary subset $X \subseteq G$, the smallest subgroup H of G such that $X \subseteq H$ is I_{Ψ_G}, where

$$\Psi_G = \{\varnothing \to x \mid x \in X \cup \{e\}\} \cup \left\{\{a, b\} \to ab^{-1} \mid a, b \in G\right\}$$

Here, e is the identity and b^{-1} is the inverse of b in G.

Example 3. The Borel sets of reals are elements of I_{Ψ_B}, where

$$\begin{aligned}\Psi_B = \quad & \{\varnothing \to X \mid X \subseteq \mathbb{R} \text{ and open } \} \cup \\ & \left\{\{A_n \mid n \in \mathbb{N}\} \to \bigcup \neg A_n \mid A_n \subseteq \mathbb{R}\right\}\end{aligned}$$

4.2. Type-theoretic approach: Calculus of Inductive Constructions

In this section we give a very short history of type theory focussed on inductive types. It is very likely that any trial to be more specific would go far beyond the scope of this paper.

It is widely known that type theory can be used as a foundation for mathematics. Indeed, Russel presented type theory as such in his 1908 paper. Interestingly, Russell's paper appeared the same year as [Zermelo 1908] which presents set theory as a foundation for mathematics.

The type structure in Russell's *Principia Mathematica* is elegantly represented in [Church 1940] based on λ-calculus. Church's λ-calculus provides a general notation for functions:

$M ::= x \mid M\,M \mid \lambda x.M$

- every variable is a function symbol;
- every juxtaposition of two function symbols is a function symbol;
- every $\lambda x.M$ is a function symbol;
- there are no other function symbols.

Curry had similar ideas before the publication of Church's paper as is well presented in [Hindley 1997]. Curry's idea is to look at types as predicates over lambda terms, writing $M: A$ to express that M satisfies the type A.

Gödel used an extended type theory, called Gödel's T, in his *Dialectica interpretation* (Gödel 1958) to show the consistency of first-order arithmetic. The system T is equipped with an extra term constructor to deal with the principle of induction in the Peano arithmetic.

On the other hand, Girard's System F in [Girard 1972] contains no extra term constructor for recursor, but can *internally* build it. The System F is a type system with polymorphism.[7)] Syntactically, the system is very simple and contains symbols only for implication and universal quantification. However, Girard showed that his system is an extension of Gödel's T and that second-order arithmetic can be interpreted in his system. In particular, all the logical connectives, natural numbers, and a

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7) Girard proved the consistency of his system using the reducibility method, that had been introduced by [Tait 1967] while analysing [Gödel 1958]. A similar system was introduced independently by [Reynolds 1974] while analysing the notion of polymorphism in computer science.

recursor can be encoded nicely as is well demonstrated in [Girard et al. 1989].

Martin-Löf's type theory (MLTT in short, [Martin-Löf 1975] and [Martin-Löf 1980]) is a constructive formalism of type theory equipped with a facility to deal with inductive definitions. MLTT with W-types, the type of well-founded trees, can build the notion of inductive types as a derived notion.

People, however, recognized that it is more elegant to have the notion of inductive types in the core of the formal system itself, rather to build it as a derived notion. [Coquand and Paulin-Mohring 1989] introduces an extension of Church's simple type theory where inductively defined types are added. Later, [Pfenning and Pauling-Mohring 1990] uses another approach to extend Coquand's Calculus of Constructions (Coquand 1985) to deal with inductively defined types in a more simple and elegant style. They showed that all primitive recursive functionals over the inductively defined types can also be represented in their system. This system is further refined to the *Calculus of Inductive Constructions* and used as the underlying formal language of the proof assistant *Coq*, cf. [Paulin-Mohring 1997].

Coq is a computer software which allows to express mathematical assertions, mechanically checks proofs of these assertions, and helps to find formal proofs. Further information about *Coq* can be found in its homepage.[8] Here we just demonstrate how to inductively define the type of natural numbers in *Coq*:

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8) https://coq.inria.fr

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Inductive nat : Set :=
| 0 : nat
| S : nat → nat.
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The word Inductive is a keyword denoting that nat is the type of objects which are going to be defined inductively. The type nat is supposed to be the type of objects which can be used as natural numbers. There are two constructors to build the objects of the inductive type nat. The first one is 0, representing the number zero. And it is directly declared to be an object of type nat. The second constructor S stands for the successor function and is of type nat to nat: Given an arbitrary object n of type nat, S n is another object of type nat.

The aforementioned example is, of course, a very simple one. Here we just mention that *Coq* is capable of much more. In fact, it is at least as strong as the Zermelo-Fraenkel set theory with very big cardinals. Further information about the strength of *Coq* can be found in [Werner 1997]. [Lee and Werner 2011] explains how to set-theoretically understand inductive types in *Coq*.

5. Predicative vs. impredicative definition of numbers

In this section we discuss Frege's use of impredicativity in the definition of natural numbers. Note first that the strong ancestral R^+ of a relation R is the same as the transitive closure R^T of R. To see this, we first recall the definition of the two concepts.

First, given a (binary) relation R, the strong ancestral is defined as follows:

$$R^{+}(x,y) := \forall F [\forall z (R(x, z) \rightarrow Fz) \wedge Her(F,R) \rightarrow Fy]$$

where

$$Her(F,R) := \forall x\, y\, (R(x, y) \wedge Fx \rightarrow Fy) \ .$$

Secondly, the transitive closure R^T is the smallest transitive relation containing R. The transitive closure R^T always exists and can be defined as follows:

$$R^T := \bigcup_{i \geqq 1} R^i \ ,$$

where R^i is inductively defined by

$$R^1(a, b) \leftrightarrow R(a, b) \quad , \text{ and}$$
$$R^{i+1}(a, b) \leftrightarrow \exists c\, [R^i(a, c) \wedge R(c, b)] \quad .$$

It is obvious that R^+ is transitive and contains R. Moreover, R^T contains R^+: Assume $R^+(a, b)$ holds. We should show that $R^T(a, b)$ holds, too. For this, take $Fx \equiv R^T(a, x)$. Then the premisses $\forall z (R(x, z) \rightarrow Fz)$ and $Her(F,R)$ of $R^+(a, b)$ with $Fx \equiv R^T(a, x)$ hold obviously. So $R^T(a, b)$ holds.

This proof of the equivalence of the two concepts ‘ancestral’ and ‘transitive closure’ implies that Frege’s concept of ancestral, hence the definition of natural numbers are all impredicative. Note that impredicativity is one of main factors in Frege’s work. It would be hardly possible to reconstruct Frege's work without it. More about Frege's use of impredicativity we refer to [Ferreira 2005] and more about impredicativity in general to [Feferman 2005].

This kind of impredicative definition of natural numbers is also used in Girard's System F, cf. [Girard et al. 1989]. On the other hand, Gödel's T and the Calculus of Inductive Constructions provide facilities to deal with natural numbers in a predicative way as demonstrated in the previous section.

[Dummett 1991] asks in the opening of Chapter 17: "How did the serpent of inconsistency enter Frege's paradise?" He blames impredicativity in Frege's system as the main cause of its inconsistency. Although Dummett's blame may be unfair for impredicativity[9], his view is later in some way confirmed by [Heck 1996] who introduces a consistent, ramified predicative second-order fragment of Frege's system. See also [Wehmeier 1999] how one can extend Heck's predicative system in a consistent way.

6. Conclusion

Frege's system in *Grundgesetze* is basically second-order logic augmented by a single non-logical axiom called *Axiom V*. Despite the inconsistency, the *Grundgesetze* contains all the essential steps necessary to prove the fundamental propositions of arithmetic,

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9) Unfair in the sense that systems like System F and Calculus of Inductive Constructions are consistent impredicative systems.

including the principle of induction on natural numbers, from a single principle known as *Hume's Principle*. We discussed how Frege realized in a purely logical way the concept of natural numbers and the principle of induction. Moreover, we also illustrated the impact of his understanding of natural numbers and induction on the development of the modern logic systems both in set theory and type theory.

It would be interesting to analyze Dummett’s blame on impredicativity for the inconsistency of Frege’s system in *Grundgesetze*. Although some answers are already provided, an ultimate answer is still missing.

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프레게의 Grundgesetze 와 그 이후의 시스템에서의 귀납법 고찰

이 계 식

프레게의 Grundgesetze에 소개된 시스템과 그 이후에 집합론 및 유형론에서 중요한 역할을 한 시스템들에서 사용된 귀납법에 대해 살펴본다. 먼저 프레게의 자연수 귀납법에 대한 이해를 살펴 본 후에 현대 집합론과 유형론에서 귀납법이 어떻게 정의 및 활용되는가를 살펴본다. 또한 프레게의 접근방식과 기타 접근방식의 차이점을 predicativity와 impredicativity 차원에서 조명한다.

주요어: 귀납법, 프레게, Grundgesetze, 집합론, 유형론