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Routley-Meyer semantics for R*

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【Abstract】This paper deals with Routley-Meyer semantics for two versions of **R** of Relevance. For this, first, we introduce two systems $\mathbf{R^t}$, $\mathbf{R^T}$ and their corresponding algebraic semantics. We next consider Routley-Meyer semantics for these systems.

【Key Words】Routley-Meyer semantics, algebraic semantics, Kripke-style semantics, **R**, $\mathbf{R^0}$, $\mathbf{R^t}$, $\mathbf{R^T}$.

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1. Introduction

Kripke-style semantics are known as binary relational semantics for modal and intuitionistic logics (Kripke (1963; 1965a; 1965b). But, in general, this semantics does not work for relevance logics (see Dunn (1986)). Because of this, Routley and Meyer introduced the so-called Routley-Meyer semantics for relevance logics (see Routley and Meyer (1972; 1973)). This semantics is a generalization of Kripke-style semantics to ternary relational semantics. So far, many logicians have had difficulties in providing Kripke-style semantics for relevance logics. Recently, Yang provided Kripke-style semantics (as well as algebraic semantics) for **R** of Relevance (Yang (2014)).

The aim of this paper is to provide Routley-Meyer semantics for **R**. To some readers this seems strange because, as mentioned above, Routley-Meyer semantics is known to us as semantics for relevance logics, in particular for **R**. However, as Yang noted in his (2013), there are at least three versions of **R**. One is the system $\mathbf{R}^0$ that has no propositional constants; another is the system $\mathbf{R}^\mathbf{t}$ that has propositional constants **t**, **f**; the other is the system $\mathbf{R}^\mathbf{T}$ that has propositional constants **t**, **f**, **T**, **F**. The well-known Routley-Meyer semantics for **R** is that for $\mathbf{R}^0$ but not for $\mathbf{R}^\mathbf{t}$ and $\mathbf{R}^\mathbf{T}$ (see Dunn (1986)).

Here, we introduce Routley-Meyer semantics for the other two versions of **R**, i.e., $\mathbf{R}^\mathbf{t}$ and $\mathbf{R}^\mathbf{T}$. One interesting fact is that Routley-Meyer semantics, which will be introduced here, does not require star operation * for negation. Note that, in general, Routley-Meyer semantics requires that operation for negation. Thus, our semantics can be regarded as *Routley-Meyer semantics without star operation* *.

This paper is organized as follows. In Sect. 2, we introduce the systems $\mathbf{R}^{\mathbf{t}}$ and $\mathbf{R}^{\mathbf{T}}$, along with their corresponding algebraic semantics. In Sect. 3, we provide Routley-Meyer semantics for these systems. We prove that $\mathbf{R}^{\mathbf{t}}$ and $\mathbf{R}^{\mathbf{T}}$ are sound and complete with respect to (w.r.t.) such semantics.

For convenience, we adopt the notations and terminology similar to those in Anderson, Belnap, & Dunn (1992), Dunn (1986), Dunn & Hardegree (2001), Yang (2013, 2014), and assume reader familiarity with them (together with results found therein).

2. Two versions of R: $\mathbf{R}^{\mathbf{t}}$ and $\mathbf{R}^{\mathbf{T}}$

In this section, we introduce two versions of **R** $\mathbf{R}^{\mathbf{t}}$ and $\mathbf{R}^{\mathbf{T}}$. We base $\mathbf{R}^{\mathbf{t}}$ on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\to$, $\wedge$, $\vee$, and a constant **f**, with defined connectives:[1)]

df1. $\sim\phi := \phi \to \mathbf{f}$

df2. $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$

df3. $\phi \,\&\, \psi := \sim(\phi \to \sim\psi)$.

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1) Note that, while $\wedge$ is the extensional conjunction connective, & is the intensional conjunction one.

The constant **t** is defined as **f** $\to$ **f**. We moreover define $\phi_t := \phi \wedge \mathbf{t}$. For the remainder, we shall follow the customary notations and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatizations of $\mathbf{R}^{\mathbf{t}}$ and $\mathbf{R}^{\mathbf{T}}$.

Definition 2.1 (Yang (2013))

(i) $\mathbf{R}^{\mathbf{t}}$ consists of the following axiom schemes and rules:

A1. $\phi \to \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi)$ ($\vee$-elimination, $\vee$-E)

A6. $(\phi \wedge (\psi \vee \chi)) \to ((\phi \wedge \psi) \vee (\phi \wedge \chi))$ ($\wedge\vee$-distributivity, $\wedge\vee$-D)

A7. $\phi \leftrightarrow (\mathbf{t} \to \phi)$ (push and pop, PP)

A8. $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ (suffixing, SF)

A9. $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$ (residuation, RE)

A10. $(\phi \to (\phi \to \psi)) \to (\phi \to \psi)$ (contraction, CR)

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj).

(ii) $\mathbf{R}^{\mathbf{T}}$ is an axiomatic expansion of $\mathbf{R}^{\mathbf{t}}$ with constant **F**, and its corresponding axiom scheme:

A11. $\mathbf{F} \to \phi$.

Note that $\phi \to \psi$ can be defined as $\sim(\phi \,\&\, \sim\psi)$ (df4) in L ($\in \{\mathbf{R}^{\mathbf{t}}, \mathbf{R}^{\mathbf{T}}\}$). Note also that **T** is defined as ~**F** in $\mathbf{R}^{\mathbf{T}}$.

Proposition 2.2 (i) L ($\in$ {$\mathbf{R}^t$, $\mathbf{R}^T$}) proves:

(1) $(\phi \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi \,\&\, \psi) \,\&\, \chi)$ (&-associativity, AS)

(2) $(\phi \,\&\, \psi) \rightarrow (\psi \,\&\, \phi)$ (&-commutativity, &-C)

(3) $\phi \rightarrow (\phi \,\&\, \phi)$ (contraction2, CR2)

(4) $(\phi \wedge \psi) \rightarrow (\phi \,\&\, \psi)$

(5) $(\phi \,\&\, \mathbf{t}) \leftrightarrow \phi$

(6) $(\phi \rightarrow {\sim}\phi) \rightarrow {\sim}\phi$ (reductio, RD)

(7) $(\phi \rightarrow \psi) \rightarrow ({\sim}\psi \rightarrow {\sim}\phi)$ (contraposition, CP)

(8) ${\sim}{\sim}\phi \leftrightarrow \phi$ (double negation, DN).

(ii) $\mathbf{R}^T$ proves:

(1) $\phi \rightarrow \mathbf{T}$.

Proof: (i) For (1) to (4), see Anderson & Belnap (1975).

The left-to-right direction of (5) follows from A8, df2, A2, and A10. For the right-to-left direction of (5), let $(\phi \,\&\, \mathbf{t}) \rightarrow (\phi \,\&\, \mathbf{t})$ by A1. Then, we have $\mathbf{t} \rightarrow (\phi \rightarrow (\phi \,\&\, \mathbf{t}))$ by A9 and (2); therefore, $\phi \rightarrow (\phi \,\&\, \mathbf{t})$ by A1, df1, and (mp).

(6) follows from A10 and df1.

(7) follows from A8 and df1.

The left-to-right direction of (8) follows from (5), df2, A2, and df3. For the right-to-left direction of (8), let $(\phi \rightarrow \mathbf{f}) \rightarrow (\phi \rightarrow \mathbf{f})$ by A1. Then, we obtain $\phi \rightarrow ((\phi \rightarrow \mathbf{f}) \rightarrow \mathbf{f})$ by A9 and (2); therefore, $\phi \rightarrow {\sim\sim}\phi$ by df1.

(ii) (1) follows from A11, (i) (7), and (mp). □

Note that the system $\mathbf{R}^0$ requires (i) (6) to (8) in Proposition 2 as the axioms for negation (see Dunn (1986)). Thus, we can say that all the negation axioms for $\mathbf{R}^0$ are provable in $\mathbf{R}^t$ and $\mathbf{R}^T$.

A *theory* over L ($\in$ {$\mathbf{R}^t$, $\mathbf{R}^T$}) is a set T of formulas. A *proof* in a theory T over L is a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using the two rules in Definition 2.1. T $\vdash$ ϕ, more exactly T $\vdash_L$ ϕ, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. The relevant deduction theorem (RDT_t) for L is as follows:

Proposition 2.3 (Meyer, Dunn, & Leblanc (1976)) Let T be a theory, and ϕ, ψ formulas.

(RDT_t) T $\cup$ {ϕ} $\vdash$ ψ if and only if (iff) T $\vdash$ $\phi_t \to \psi$.

For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\to$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

The algebraic counterpart of L is the class of *L-algebras*. Let $x_t := x \wedge t$. They are defined as follows.

Definition 2.4 (i) A *pointed commutative residuated distributive lattice* is a structure $\mathbf{A}$ = (A, t, f, $\wedge$, $\vee$, *, $\to$) such that:

(Ⅰ) (A, $\wedge$, $\vee$) is a distributive lattice.

(Ⅱ) (A, *, t) is a commutative monoid.

(Ⅲ) $y \leq x \to z$ iff $x * y \leq z$, for all x, y, z $\in$ A (residuation).

(ii) A *pointed bounded commutative residuated distributive lattice* is a pointed commutative residuated distributive lattice satisfying:

(I') (A, $\wedge$, $\vee$, $\top$, $\bot$) is a bounded distributive lattice, where $\top$ and $\bot$ are top and bottom elements.

(iii) (Dunn-algebras, Anderson & Belnap (1975), Anderson, Belnap, & Dunn (1992)) A *Dunn-algebra* is a pointed commutative residuated distributive lattice satisfying:

(IV) $x \leq x * x$ (contraction).

(V) $(x \rightarrow f) \rightarrow f \leq x$ (double negation elimination).

(iv) (R^T-algebras) An *R^T-algebra* is a Dunn-algebra satisfying (I').

We call Dunn-algebras R^t-algebras because the class of Dunn-algebras characterizes the system $\mathbf{R}^{\mathbf{t}}$. Note that Dunn-algebras are also called De Morgan monoids. We further call all of R^t- and R^T-algebras *L-algebras*.

Additional unary and binary operations are defined as in Sect. 2.1.

The class of all L-algebras is a variety which will be denoted by $\mathbb{L}$.

Definition 2.5 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function v : FOR $\rightarrow \mathcal{A}$ satisfying: $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{f}) = f$, and hence $v({\sim}\phi) = {\sim}v(\phi)$ and $v(\mathbf{t}) = t$, (and $v(\mathbf{F}) = \bot$, and hence $v(\mathbf{T}) = \top$ w.r.t. $\mathbf{R}^{\mathbf{T}}$).

Definition 2.6 (Cintula (2006)) Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and $\mathbb{K}$ a class of L-algebras.

(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.
(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq t$ for each $\phi \in T$. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.
(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_{K} \phi$, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ $\{\phi\}$, $\mathcal{A}$) for each $\mathcal{A} \in$ K.

Definition 2.7 (**L**-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 2.6. $\mathcal{A}$ is an ***L**-algebra* iff whenever ϕ is L-provable in T (i.e. $T \vdash_{L} \phi$), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. $T \vDash_{\{A\}} \phi$), $\mathcal{A}$ an L-algebra. By *MOD(**L**)*, we denote the class of **L**-algebras. Finally, we write $T \vDash_{L} \phi$ in place of $T \vDash_{MOD(\mathbf{L})} \phi$.

Note that since each condition for the L-algebra has a form of equation or can be defined in equation (exercise), it can be ensured that the class of all L-algebras is a variety.

We first show that classes of provably equivalent formulas form an L-algebra. Let T be a fixed theory over L ($\in \{\mathbf{R^t}, \mathbf{R^T}\}$). For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash_{L} \phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \rightarrow [\psi]_T = [\phi \rightarrow \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \,\&\, \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $t = [\mathbf{t}]_T$, $f = [\mathbf{f}]_T$, (and $\top = [\mathbf{T}]_T$ and $\bot = [\mathbf{F}]_T$ w.r.t. $\mathbf{R^T}$.) By $\boldsymbol{A}_T$, we denote this algebra.

Proposition 2.8 For T a theory over L, $\mathbf{A}_T$ is an **L**-algebra.

Proof: For the fact that $\mathbf{A}_T$ (T over $\mathbf{R}^t$) is an $\mathbf{R}^t$-algebra, see Proposition 2.8 in Yang (2012). In order to show that $\mathbf{A}_T$ (T over $\mathbf{R}^\mathbf{T}$) is an $\mathbf{R}^\top$-algebra, we just note that: $[\phi]_T \leq [\mathbf{T}]_T$ iff $T \vdash_{\mathbf{R}^\mathbf{T}} \phi \leftrightarrow (\phi \wedge \mathbf{T})$ iff $T \vdash_{\mathbf{R}^\mathbf{T}} \phi \rightarrow \mathbf{T}$ and $[\mathbf{F}]_T \leq [\phi]_T$ iff $T \vdash_{\mathbf{R}^\mathbf{T}} \mathbf{F} \leftrightarrow (\phi \wedge \mathbf{F})$ iff $T \vdash_{\mathbf{R}^\mathbf{T}} \mathbf{F} \rightarrow \phi$. Thus, it is an $\mathbf{R}^\top$-algebra. □

Theorem 2.9 (Strong completeness) Let T be a theory, and ϕ a formula. $T \vdash_L \phi$ iff $T \vDash_L \phi$.

Proof: The left-to-right direction follows from definition. The right-to-left direction is as follows: from Proposition 2.8, we obtain $\mathbf{A}_T \in$ MOD(L), and for $\mathbf{A}_T$-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in$ Mod(T, $\mathbf{A}_T$). Thus, since from $T \vDash_L \phi$ we obtain that $[\phi]_T = v(\phi) \geq t$, $T \vdash_L \mathbf{t} \rightarrow \phi$. Then, since $T \vdash_L \mathbf{t}$, by (mp) $T \vdash_L \phi$, as required. □

3. Routley-Meyer semantics for two versions of R

Here, we consider Routley-Meyer semantics for L ($\in \{\mathbf{R}^t, \mathbf{R}^\mathbf{T}\}$). Following Anderson, Belnap, & Dunn (1992), Dunn (1986), and Dunn & Hardegree (2001), calling relevant model structures *Routley-Meyer (RM) frames*, we define an *(RM) frame*. A frame is a structure $\mathbf{S} = (U, \sqsubseteq, R, Z)$, where $(U, \sqsubseteq, R, Z)$ is a left assertional frame[2] such that the following definitions and

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[2] That is, U is a set, Z ($\subseteq$ U) is a left lower identity ($Z \circ A \subseteq A$)

postulates hold:[3] ($\zeta \in Z$)

df5. $\alpha \sqsubseteq \beta := \exists\zeta(R\zeta\alpha\beta)$
df6. $R^2\alpha\beta\gamma\delta := \exists\chi(R\alpha\beta\chi \ \& \ R\chi\gamma\delta)$
df7. $R^2\alpha(\beta\gamma)\delta := \exists\chi(R\alpha\chi\delta \ \& \ R\beta\gamma\chi)$

(W.r.t. the following postulates, just for convenience, to represent some ζ we take ***0***, which Routley and Meyer take in their semantics. Note that ***0***, by which we represent some ζ ($\in Z$), itself is a member of Z, i.e., ***0*** $\in Z$.[4])

p0. $R\alpha\beta\gamma$ and $\alpha' \sqsubseteq \alpha$ imply $R\alpha'\beta\gamma$ (monotonicity)
p1. $R\boldsymbol{0}\alpha\alpha$
p2. $R^2\alpha\beta\gamma\delta \Rightarrow R^2\alpha(\beta\gamma)\delta$
p3. $R\alpha\beta\gamma \Rightarrow R\beta\alpha\gamma$

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satisfying the following lli
(lli) $\exists\zeta, \in Z, (R\zeta\alpha\beta)$ iff $\alpha \sqsubseteq \beta$,
$R \subseteq U^3$, and $\sqsubseteq$ is a partial-order satisfying:
$R\alpha\beta\gamma$ & $\alpha' \sqsubseteq \alpha$ imply $R\alpha'\beta\gamma$,
$R\alpha\beta\gamma$ & $\beta' \sqsubseteq \beta$ imply $R\alpha\beta'\gamma$,
$R\alpha\beta\gamma$ & $\gamma' \sqsubseteq \gamma$ imply $R\alpha\beta\gamma'$.
More exactly to understand a left assertional frame, see Dunn & Hardegree (2001). Note that U is expressed as K in Dunn (1986) (as well as in Routley & Meyer (1972; 1973); and that, for convenience, we take a left lower identity instead of a right lower one, which Dunn and Hardegree take in their (2001).

3) Note that we take df5 for the modal character of E (see Anderson, Belnap, & Dunn (1992)).

4) Often, in proofs of Sects. 4 and 5, by ***0*** we shall also ambiguously represent some ζ, if we do not need distinguish them, but context should determine what is intended.

p4. $R\alpha\alpha\alpha$ (idempotence)

Note that the system $\mathbf{R}^0$ does not have propositional constants **t** and **f** and so the negation ~ is not definable in $\mathbf{R}^0$. Thus, for $\mathbf{R}^0$ we need not only the postulates p0 to p4, but also

p5. $R\alpha\beta\gamma \Rightarrow R\alpha\gamma^*\beta^*$ and
p6. $\alpha^{**} = \alpha$ (see Dunn (1986)).

As the results below will show, it suffices to have the postulates p0 to p4 for L ($\in \{\mathbf{R}^t, \mathbf{R}^T\}$). Following Dunn (and Hardegree) (2000) (and (2001)), we regard U as a set of "states of information," and for α, $\beta \in$ U, $\alpha \sqsubseteq \beta$ means that the information of α is included in that of β.

By a *model* for L, we mean a structure **M** = (U, $\sqsubseteq$, R, Z, $\vDash$), where (U, $\sqsubseteq$, R, Z) is a frame and $\vDash$ is a relation from U to sentences of L ($\in \{\mathbf{R}^t, \mathbf{R}^T\}$) satisfying the following conditions:

(Atomic Hereditary Condition (AHC))
for a propositional variable p, if $\alpha \vDash p$ and $\alpha \sqsubseteq \beta$, then $\beta \vDash p$;

(Evaluation Clauses (EC)) for formulas ϕ, ψ
($\wedge$) $\alpha \vDash \phi \wedge \psi$ iff $\alpha \vDash \phi$ and $\alpha \vDash \psi$;
($\vee$) $\alpha \vDash \phi \vee \psi$ iff $\alpha \vDash \phi$ or $\alpha \vDash \psi$;
($\rightarrow$) $\alpha \vDash \phi \rightarrow \psi$ iff for all β, $\gamma \sqsupseteq \alpha$, if $R\alpha\beta\gamma$ and $\beta \vDash \phi$, then $\gamma \vDash \psi$.

((**F**) $\alpha \vDash$ **F** never for $\mathbf{R}^{\mathrm{T}}$.)

A formula ϕ is *true* on v at α of U just in case $\alpha \vDash \phi$; ϕ is *verified* on **M** in case ζ (especially ***0***), $\in$ Z, $\vDash \phi$; ϕ *entails* ψ on **M** in case $\forall \chi \in$ U, if $\chi \vDash \phi$, then $\chi \vDash \psi$; ϕ *L-entails* ψ just in case ϕ entails ψ in every model; and ϕ is *L-valid* in a frame **S** just in case it is verified in all evaluations therein. Let Σ be the class of frames. A sentence ϕ is L-valid, in symbols $\vDash_{L} \phi$, iff $\forall$**S** $\in \Sigma$, ϕ is L-valid in **S**.

Following Anderson, Belnap, & Dunn (1992) and Dunn (1986), we give the soundness for L. To prove it, we need the Verification Lemma below. First, by an induction on ϕ, we can easily prove the following.

Lemma 3.1 (Hereditary Condition (HC)) For any formula ϕ, if $\alpha \vDash \phi$ and $\alpha \sqsubseteq \beta$, then $\beta \vDash \phi$.

Since w.r.t. the connectives $\wedge$, $\vee$, $\rightarrow$, we have the same evaluations as in Anderson, Belnap, & Dunn (1992), Dunn (1986), Routley & Meyer (1973), we can use the Verification Lemma in them. Thus,

Lemma 3.2 (Verification Lemma) ϕ entails ψ on v only if $\phi \rightarrow \psi$ is verified, i.e., true at ζ ($\in$ Z), on v. Thus, ϕ entails ψ in a given model **M**, = (U, $\sqsubseteq$, R, Z, $\vDash$), only if $\phi \rightarrow \psi$ is L-valid in the model; that is, for every χ ($\in$ U) if $\chi \vDash \phi$ then $\chi \vDash \psi$ only if $\zeta \vDash \phi \rightarrow \psi$. And ϕ L-entails ψ only if $\phi \rightarrow \psi$ is

L-valid.

Proof: It is proved by Lemmas 2 and 3 in Routley & Meyer (1973) and definitions. (Using Lemma 1, we can also prove this, see the Verification Lemma in Anderson, Belnap, & Dunn (1992), Dunn (1986).) □

Let $\vdash_L \phi$ be the theoremhood of ϕ in L. We note that each postulate was used in Anderson, Belnap, & Dunn (1992) and Dunn (1986). Thus, the soundness for L is immediate.

Proposition 3.3 (Soundness) If $\vdash_L \phi$, then $\vDash_L \phi$.

Proof: We just prove that each instance of the axiom schemes A7 and A11 is valid in all frames, i.e., L-valid. For the other cases, see Dunn (1986).

For A7, it suffices by Lemma 3.2 (i) to assume $\alpha \vDash \phi$ and show $\alpha \vDash \mathbf{t} \rightarrow \phi$, and (ii) to assume $\alpha \vDash \mathbf{t} \rightarrow \phi$ and show $\alpha \vDash \phi$. To show these two, we first note that we obtain the postulate (p7) $R\alpha\boldsymbol{0}\alpha$ using p1 and p5.[5] Based on p7, we prove (i) and (ii). For (i), assume $\alpha \vDash \phi$. Then, we obtain $\alpha \vDash \mathbf{t} \rightarrow \phi$ using ($\rightarrow$) and p7. For (ii), assume $\alpha \vDash \mathbf{t} \rightarrow \phi$. Since $R\alpha\boldsymbol{0}\alpha$ and $\boldsymbol{0} \vDash \mathbf{t}$, we obtain $\alpha \vDash \phi$ by ($\rightarrow$).

For A11, it suffices by Lemma 3.2 to assume that $\alpha \vDash \mathbf{F}$ and show $\alpha \vDash \phi$. We may instead show that $\alpha \nvDash \mathbf{F}$ or $\alpha \vDash \phi$. Since by (**F**) $\alpha \vDash \mathbf{F}$ does not hold, it is obvious that $\alpha \nvDash \mathbf{F}$. □

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5) The postulate p7 was introduced in Routley & Meyer (1972).

We give the completeness for L by using the well-known Henkin-style proofs for modal logic, but with prime theories in place of maximal theories. To do this, we define some theories. We interpret $\vdash_L$ as the deducibility consequence relation of the logic L. By an *L-theory*, we mean a set Γ of sentences closed under deducibility, i.e., closed under (mp) and (adj); by a *prime L-theory*, a theory Γ such that if $\phi \vee \psi \in \Gamma$, then $\phi \in \Gamma$ or $\psi \in \Gamma$; and by a *trivial L theory*, the entire set of sentences of L. As Dunn states in Remark 4 in Dunn (2000), we note that an L-theory Γ contains all of the theorems of L. Thus it is what has been called a "regular theory" in the relevance logic literature. That is, by an L-theory we mean a regular L-theory. This means that Γ is never empty. In the results below, there is no role either for trivial L theories. Hence, by a "L theory" we mean a non-trivial one.

Let a *canonical L-frame* be a structure **S** = $(U_{can}, \sqsubseteq_{can}, R_{can}, Z_{can})$, where $\sqsubseteq_{can}$ is an information order on U_{can}, Z_{can} is a set of any prime L theory, i.e., ζ_{can} ($\in Z_{can}$), $Z_{can} \subseteq U_{can}$, U_{can} is the set of prime L theories extending ζ_{can}, R_{can} is R below restricted to U_{can},

(1) $R\alpha\beta\gamma$ iff for any formula φ, ψ of L, if $\phi \to \psi \in \alpha$ and $\phi \in \beta$, then $\psi \in \gamma$.

We call a frame *fitting* for L if for each axiom scheme of L the corresponding semantical postulate holds.

As we mentioned above, we take the ideas of proofs from the

Henkin-style completeness proofs. Thus, note that the base $\boldsymbol{0}_{can}$, i.e., $\boldsymbol{0}$, among ζ_{can} ($\in Z_{can}$), is constructed as a prime L-theory that excludes nontheorems of L, i.e., excludes ϕ such that $\nvdash_{L} \phi$. Note also that in proofs below, by $\boldsymbol{0}$, i.e., $\boldsymbol{0}_{can}$, we often represent ζ_{can} (as well as $\boldsymbol{0}$) if context can clarify what is intended. The partial orderedness of a canonical L-frame depends on * restricted on U_{can}. Then, first, it is obvious that

Proposition 3.4 A canonical L-frame is partially ordered.

Proposition 3.5 The canonically defined L-frame is a frame fitting for L.

Proof: It suffices to note that to prove the postulates it is enough for us to point out Theorem 1 of Sects. 48.3 and 48.6 in Anderson, Belnap, & Dunn (1992), Lemma 6 in Routley & Meyer (1972), and Lemma 13 in Routley & Meyer (1973). □

Next, we need to define an appropriate relation $\vDash$ on **S**, = (U_{can}, $\sqsubseteq_{can}$, R_{can}, Z_{can}). We define it to be that

$$\alpha \vDash \phi \quad \text{iff} \quad \phi \in \alpha.$$

However, we need to verify that this satisfies AHC and EC above. Note that since the positive part of L satisfies Definition 1 of Sect. 42.1 in Anderson, Belnap, & Dunn (1992), we can directly use Fact 1 and Fact 2 of Sect. 48.3 in Anderson, Belnap,

& Dunn (1992), which are considered for $\mathbf{R}^{0+}$, and thus we can use Theorem 2 of the same section.

Proposition 3.6 The canonically defined (U_{can}, $\sqsubseteq_{can}$, R_{can}, Z_{can}, $\vDash$) is indeed an L model.

Proof: AHC and the clauses ($\wedge$), ($\vee$), and ($\rightarrow$) for EC are by Theorem 2 of Sect. 48.3 in Anderson, Belnap, & Dunn (1992). For (**F**) in $\mathbf{R}^{T}$, we need to show $\alpha \nvDash$ **F**. This is immediate because α is a non-trivial theory and thus **F** $\notin \alpha$. □

Thus, (U_{can}, $\sqsubseteq_{can}$, R_{can}, Z_{can}, $\vDash$) is an L model. So, since, by construction, ***0*** excludes our chosen nontheorem ϕ and the canonical definition of $\vDash$ agrees with membership, we can state that for each nontheorem ϕ of L, there is an L model A in which ϕ is not ***0*** $\vDash \phi$. It gives us the (weak) completeness for L as follows.

Theorem 3.7 (Weak Completeness) If $\vDash_L \phi$, then $\vdash_L \phi$.

Next, let us prove the strong completeness for L. As $\mathbf{R}^{0+}$ in Anderson, Belnap, & Dunn (1992), we define ϕ to be an *L consequence* of a set of formulas γ iff for every L model, whenever $\alpha \vDash \psi$ for every $\psi \in \Gamma$, $\alpha \vDash \phi$, for (not just ***0*** but) all $\alpha \in U$. Let us say that ϕ is *L deducible* from Γ iff ϕ is in every L theory containing Γ. Then,

Proposition 3.8 If $\Gamma \nvdash_L \phi$, then there is a prime theory ζ such that $\Gamma \subseteq \zeta$ and $\phi \notin \zeta$.

Proof: Take an enumeration $\{\phi_n: n \in \omega\}$ of the well-formed formulas of L. We define a sequence of sets by induction as follows:

$\zeta_0 = \{\phi': \Gamma \nvdash_L \phi'\}$.
$\zeta_{i+1} = Th(\zeta_i \cup \{\phi_{i+1}\})$ if it is not the case that $\zeta_i, \phi_{i+1} \vdash_L \phi$,
ζ_i otherwise.

Let ζ be the union of all these ζ_n's. It is easy to see that ζ is a theory not containing ϕ. Also we can show that it is a prime.

Suppose toward contradiction that $\psi \vee \chi \in \zeta$ and $\psi, \chi \notin \zeta$. Then the theories obtained from $\zeta \cup \psi$ and $\zeta \cup \chi$ must both contain ϕ. It follows that there is a conjunction of members of ζ ζ' such that $\zeta' \wedge \psi \vdash_L \phi$ and $\zeta' \wedge \chi \vdash_L \phi$. Note that if $\vdash_L \phi_t \rightarrow \psi$, then $\phi \vdash_L \psi$. Then, using Proposition 2.3, we can obtain $(\zeta' \wedge \psi) \vee (\zeta' \wedge \chi) \vdash_L \phi$.; therefore, $\zeta' \wedge (\psi \vee \chi) \vdash_L \phi$ by the prefixing (as a theorem), A6, and (mp). From this we get that $\phi \in \zeta$, which is contrary to our supposition. □

Thus, by using Propositions 3.6 and 3.8, we can show its strong completeness as follows.

Theorem 3.9 (Strong Completeness) If $\Gamma \vDash_L \phi$, then $\Gamma \vdash_L \phi$.

4. Concluding remark

We investigated Routley-Meyer semantics for two versions of **R**, i.e., $\mathbf{R}^{\mathbf{t}}$ and $\mathbf{R}^{\mathbf{T}}$. We proved soundness and completeness theorems. We can also consider two versions of **RM** (**R** with mingle), i.e., $\mathbf{RM}^{\mathbf{t}}$ and $\mathbf{RM}^{\mathbf{T}}$, and provide Routley-Meyer semantics for these systems. We leave its investigation to the interested reader.

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R 위한 루트리-마미어 의미론

양 은 석

글에서 우리는 연관 논리 **R**의 두 버전을 위한 루트리-마이어 의미론을 다룬다. 이를 위하여 먼저 **R**의 두 버전 $\mathbf{R}^{\mathbf{t}}$와 $\mathbf{R}^{\mathbf{T}}$를 그리고 그것들에 상응하는 대수적 의미론을 소개한다. 다음으로 이 체계들을 위한 루트리-마미어 의미론을 제공한다.

주요어: 루트리-마이어 의미론, 크립키형 의미론, 대수적 의미론, **R**, $\mathbf{R}^{\mathbf{0}}$, $\mathbf{R}^{\mathbf{t}}$, $\mathbf{R}^{\mathbf{T}}$.