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Set-theoretical Kripke-style semantics for three-valued paraconsistent logic*

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【Abstract】This paper deals with non-algebraic Kripke-style semantics for three-valued paraconsistent logic: set-theoretical Kripke-style semantics. We first recall two three-valued paraconsistent systems. We next introduce set-theoretical Kripke-style semantics for them.

【Key Words】(Set-theoretical) Kripke-style semantics, algebraic semantics, three-valued logic, paraconsistent logic.

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1. Introduction

The aim of this paper is to introduce non-algebraic Kripke-style semantics, i.e, set-theoretical Kripke-style semantics, for three-valued paraconsistent logic. For this, note that the present author introduced two kinds of (binary) Kripke-style semantics, i.e., algebraic and non-algebraic Kripke-style semantics, for logics with pseudo-Boolean (briefly, pB) and de Morgan (briefly, dM) negations in Yang(201+). But the author did not consider such semantics for logics with weak-Boolean (briefly, wB) negations. While paraconsistent logics have in general wB negations, which are dual of pB negations such as the intuitionistic and Dummett-Gödel logics H and G have, it is not clear whether such semantics work for three-valued paraconsistent systems.

As its answer, the author also introduced algebraic Kripke-style semantics for three-valued *paraconsistent* systems in Yang(2014). However, it was an open problem to show that the other kind of binary Kripke-style semantics works for three-valued paraconsistent logic. This paper resolves the remaining problem by introducing non-algebraic set-theoretic Kripke-style semantics for such systems.

The paper is organized as follows. First, in Section 2, we introduce, more exactly recall the systems $\mathbf{IUML}^{-}_{3}$ (the $\mathbf{IUML}_{3}$ with a wB negation) and $\mathbf{G}^{wB}_{3}$ (the $\mathbf{G}_{3}$ with a wB negation in place of its pB negation) introduced in Yang(2014). Next, in Section 3, we introduce the other kind of binary relational Kripke-style semantics, non-algebraic set-theoretical Kripke-style semantics, for the above mentioned three-valued systems.

For ease, let us denote wB negation by - and dM negation by ~. Moreover, for convenience, we adopt notations and terminology similar to those in Dunn(2000), Metcalfe & Montagna(2007), Montagna & Sacchetti(2003; 2004), Yang(2012a; 2012b; 2012c) and assume reader familiarity with them (together with results found therein).

2. Three-valued paraconsistent systems

We base three-valued paraconsistent logics on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives →, &, ∧, ∨, and constants **F**, **f**, **t**, with a defined connective:

df1. A ↔ B := (A → B) ∧ (B → A).

We further define **T** and A_t as **F** → **F** and A ∧ **t**, respectively. We use the axiom systems to provide a consequence relation.

Definition 2.1 (Yang(2014))

(i) $\mathbf{IUML}^{-}_{3}$ consists of the following axiom schemes and rules:

df2. -A := (**T** → A) → **F**

A1. A → A (self-implication, SI)

A2. (A ∧ B) → A, (A ∧ B) → B (∧-elimination, ∧-E)

A3. ((A→B) ∧ (A→C)) → (A → (B∧C)) (∧-introduction, ∧-I)

A4. A → (A ∨ B), B → (A ∨ B) (∨-introduction, ∨-I)

A5. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$ ($\vee$-elimination, $\vee$-E)

A6. $(A \& B) \to (B \& A)$ (&-commutativity, &-C)

A7. $(A \& \mathbf{t}) \leftrightarrow A$ (push and pop, PP)

A8. $\mathbf{F} \to A$ (ex falsum quodlibet, EF)

A9. $A \to \mathbf{T}$ (verum ex quolibet, VE)

A10. $(A \to (B \to C)) \leftrightarrow ((A \& B) \to C)$ (residuation, RE)

A11. $(A \to B) \to ((B \to C) \to (A \to C))$ (suffixing, SF)

A12. $(A \to B)_{\mathbf{t}} \vee (B \to A)_{\mathbf{t}}$ (**t**-prelinearity, $PL_{\mathbf{t}}$)

A13. $\sim\sim A \to A$ (double negation elimination, DNE)

A14. $(A \& A) \leftrightarrow A$ (idempotence, ID)

A15. $\mathbf{t} \leftrightarrow \mathbf{f}$ (fixed-point, FP)

A16. $A \to (\sim A \to A)$ (RM3(1))

A17. $A \vee (A \to B)$ (RM3(2))

A18. $--A \to A$ (classical double negation, ClDN)

A19. $A \to (B \vee -B)$ (triviality, TRI)

A20. $(A \to B) \to (-B \to -A)$ (contraposition, CP^-)

A21. $(A \wedge -B) \to -(A \to B)$ (-1)

A22. $\sim A \to -A$ (-2)

A23. $-(A \& B) \to ((A \wedge B) \to (-A \wedge -B))$ (-3)

A24. $--(A \& B) \to (-A \to B)$ (-4)

A25. $((A \to B) \wedge -(A \to B)) \to (A \wedge -B)$ ($IUML^-3$)

$A \to B, A \vdash B$ (modus ponens, mp)

$A, B \vdash A \wedge B$ (adjunction, adj)

(ii) $\mathbf{G}^{wB}{}_{3}$ is A1 - A12, A14, A18, A19, (mp), (adj) plus

A26. $A \to (B \to A)$ (weakening, W)

A27. $-(A \wedge B) \leftrightarrow (-A \vee -B)$ ($DM1^-$)

A28. $-(A \vee B) \leftrightarrow (-A \wedge -B)$ ($DM2^-$)

A29. $((A \to -(C \vee -C)) \to B) \to (((B \to A) \to B) \to B)$ (G3)
A30. $((A \to B) \wedge -(A \to B)) \to (--A \wedge -B)$ ($G^{-}3(1)$)
A31. $(--A \wedge -B) \to -(A \to B)$ ($G^{-}3(2)$)

For easy reference, we let Ls_3 be the set of the three-valued systems introduced in Definition 2.1.

Definition 2.2 $Ls_3 = \{\mathbf{IUML}^{-}_3, \mathbf{G}^{wB}_3\}$.

A *theory* is a set of formulas closed under consequence relation. A *proof* in a theory Γ over L_3 ($\in Ls_3$) is a sequence *s* of formulas such that each element of s is either an axiom of L_3, a member of Γ, or is derivable from previous elements of s by means of a rule of L_3. $\Gamma \vdash A$, more exactly $\Gamma \vdash_{L_3} A$, means that A is *provable* in Γ with respect to (w.r.t.) L_3, i.e., there is an L_3-proof of A in Γ. A theory Γ is *trivial* if $\Gamma \vdash \mathbf{F}$; otherwise, it is *non-trivial*.

The deduction theorems for L_3 are as follows:

Proposition 2.3 (Yang(2014)) Let Γ be a theory over L_3 and A, B be formulas.

(i) $\Gamma \cup \{A\} \vdash_{\mathbf{IUML}^{-}_3} B$ iff $\Gamma \vdash_{\mathbf{IUML}^{-}_3} A_{\mathbf{t}} \to B$.

(ii) $\Gamma \cup \{A\} \vdash_{\mathbf{G}^{wB}_3} B$ iff $\Gamma \vdash_{\mathbf{G}^{wB}_3} A \to B$.

The following formulas can be proved straightforwardly.

Proposition 2.4 (Yang(2014))

(i) L_3 ($\in Ls_3$) proves:

(1) (A & (B & C)) → ((A & B) & C) (associativity, AS)

(2) (A → B) ∨ (B → A) (prelinearity, PL)

(3) A ∨ -A (excluded middle, EM)

(ii) $\mathbf{IUML}^{-}_{3}$ proves:

(1) ~~ A ↔ A (double negation, DN)

(iii) $\mathbf{G}^{wB}_{3}$ proves (CP^{-}) and:

(1) **t** ↔ **T** (INT).

3. Set-theoretical Kripke-style semantics

3.1. Semantics

Here, we consider non-algebraic set-theoretical and binary relational Kripke-style semantics for L_3. Let us regard an evaluation to be a function from sentences to non-empty sets of two truth values, including the set having both truth values to account for overdetermination. We regard a three-valued matrix as a lattice and call it the *lattice* $\mathbf{3}_B$; we denote each set of value(s) {0}, {1}, and {0, 1} by F, T, and B, respectively (see Figure 1).

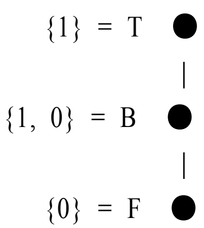

Figure 1: The lattice $\mathbf{3}_B$

Each matrix for ~, -, ∧, ∨, and → can be defined as in Table 1 (+ indicates the designated value(s)).[1)]

-	
T+	F
B	T
F	T

~	
T+	F
B+	T
F	T

∧	T	B	F
T+	T	B	F
B(+)	B	B	F
F	F	F	F

∨	T	B	F
T+	T	T	T
B(+)	T	B	B
F	T	B	F

$\rightarrow_{G3}$	T	B	F
T+	T	B	F
B	T	T	F
F	T	T	T

$\rightarrow_{RM3}$	T	B	F
T+	T	F	F
B+	T	B	F
F	T	T	T

Table 1: Three-valued matrices for evaluations of L_3

Note that, in Table 1, we take $\rightarrow_{G3}$ and $\rightarrow_{RM3}$ for $\mathbf{G}^{wB}_3$ and $\mathbf{IUML}^{-}_3$, respectively.

Next, as in Dunn(2000), let us define evaluations. An *evaluation* into $\mathbf{3}_B$ is a function v from sentences into $\mathbf{3}_B$ such that v(-A) =

1) We do not have to introduce the matrix for & because & is ∧ in $\mathbf{G}^{wB}_3$, and definable in $\mathbf{IUML}^{-}_3$ using ~ and → connectives. Note that, while the matrices for $\mathbf{G}^{wB}_3$ have one desiganted element T, the mattrices for $\mathbf{IUML}^{-}_3$ have the two T, B. By *(+)*, we ambiguously express these in the matrices for ∧ and ∨.

-v(A), v(~A) = ~v(A), v(A $\wedge$ B) = v(A) $\wedge$ v(B), v(A $\vee$ B) = v(A) $\vee$ v(B), and v(A $\rightarrow$ B) = v(A) $\rightarrow$ v(B). As the labeling of Figure 1 reveals, we can view $\mathbf{3}_B$ as consisting of subsets of the usual two truth values. Thus, equivalently, an evaluation can be regarded as a map v from sentences into the powerset of {1, 0} (see below). For a *total evaluation*, we always have at least one of 0, 1 $\in$ v(A). We write $\Vdash^v_1$ A for 1 $\in$ v(A) and $\Vdash^v_0$ A for 0 $\in$ v(A). Like the two-valued matrix for classical logic CL, we call a matrix *characteristic* for a calculus L when a formula A is provable if it assumes a designated value for every assignment of values to its variables, i.e., if L is weak complete w.r.t. the matrix (see e.g. Dunn(2000) and Dunn & Hardegree(2001).

Definition 3.1 (Dunn(2000)) A *binary relational Kripke frame* (briefly a *frame*) is a structure **S** = (U, ζ, $\sqsubseteq$), where $\zeta \in$ U and $\sqsubseteq$ is a partial order (p.o.) on U.}

As X in Section 3, we regard U as a set of nodes. Then, ζ is the base state of information, and it further does not hurt to require that ζ be the least element of U under $\sqsubseteq$. By Σ, we denote the class of all frames. For L_3, we need to consider frames where $\sqsubseteq$ is connected in the sense that, for any α, $\beta \in$ U, either $\alpha \sqsubseteq \beta$ or $\beta \sqsubseteq \alpha$. A *linear order* (l.o.) is a connected partial order. Then a *linear frame* is a structure **S** = (U, ζ, $\sqsubseteq$), where $\zeta \in$ U and $\sqsubseteq$ is an l.o. on U.

We assume that there are denumerably many atomic sentences, and that the class of formulas *Fm* is defined inductively from these in the usual manner, utilizing the connectives -, ~, $\wedge$, $\vee$, and $\rightarrow$. A (parameterized) *L_3-evaluation* on a linear frame **S** is a function v(A, α) from *Fm* × U into $\mathbf{3}_B$ subject to the conditions below. We denote the set of these evaluations as $\mathbf{Val}_{L3}$, and we write $\alpha \Vdash^v_1 A$ for 1 in v(A, α) and $\alpha \Vdash^v_0 A$ for 0 in v(A, α). In context, we often leave the superscript v implicit.

(Atomic Hereditary Conditions (AHC)) for any atomic sentence p,
(HC_1) $\alpha \Vdash^v_1 p$ and $\alpha \sqsubseteq \beta \Rightarrow \beta \Vdash^v_1 p$;
(HC_0) $\alpha \Vdash^v_0 p$ and $\alpha \sqsubseteq \beta \Rightarrow \beta \Vdash^v_0 p$.

The truth and falsity conditions for propositional constants **t**, **f**, **T**, **F**, and compound sentences are then given by the following clauses:

(tf_1) $\alpha \Vdash_1 \mathbf{t} \Leftrightarrow \alpha \Vdash_1 \mathbf{f}$;
(tf_0) $\alpha \Vdash_0 \mathbf{t} \Leftrightarrow \alpha \Vdash_0 \mathbf{f}$;
($\top_1$) $\alpha \Vdash_1 \mathbf{T}$ always;
($\top_0$) $\alpha \Vdash_0 \mathbf{T}$ never;
($\bot_1$) $\alpha \Vdash_1 \mathbf{F}$ never;
($\bot_0$) $\alpha \Vdash_0 \mathbf{F}$ always;
($-_1$) $\alpha \Vdash_1 -A \Leftrightarrow \alpha \Vdash_0 A$;
($-_0$) $\alpha \Vdash_0 -A \Leftrightarrow \alpha \nVdash_0 A$;
($\sim_1$) $\alpha \Vdash_1 {\sim}A \Leftrightarrow \alpha \Vdash_0 A$;
($\sim_0$) $\alpha \Vdash_0 {\sim}A \Leftrightarrow \alpha \Vdash_1 A$;
($\wedge_1$) $\alpha \Vdash_1 A \wedge B \Leftrightarrow \alpha \Vdash_1 A$ and $\alpha \Vdash_1 B$;
($\wedge_0$) $\alpha \Vdash_0 A \wedge B \Leftrightarrow \alpha \Vdash_0 A$ or $\alpha \Vdash_0 B$;
($\vee_1$) $\alpha \Vdash_1 A \vee B \Leftrightarrow \alpha \Vdash_1 A$ or $\alpha \Vdash_1 B$;

($\vee_0$) $\alpha \Vdash_0 A \vee B \Leftrightarrow \alpha \Vdash_0 A$ and $\alpha \Vdash_0 B$;

($\rightarrow_1$) $\alpha \Vdash_1 A \rightarrow B \Leftrightarrow$ (i) for all $\beta \sqsupseteq \alpha$, ($\beta \Vdash_1 A \Rightarrow \beta \Vdash_1 B$), and

(ii) for all $\beta \sqsupseteq \alpha$, ($\beta \Vdash_0 B \Rightarrow \beta \Vdash_0 A$);

($\rightarrow_{0G3}$) $\alpha \Vdash_0 A \rightarrow B \Leftrightarrow$ (i) $\alpha \Vdash_0 - A$, i.e., for all $\beta \sqsupseteq \alpha$, $\beta \nVdash_0 A$, and $\alpha \Vdash_0 B$, or

(ii) $\alpha \nVdash_1 A \rightarrow B$;

($\rightarrow_{0RM3}$) $\alpha \Vdash_0 A \rightarrow B \Leftrightarrow$ (i) $\alpha \Vdash_1 A$ and $\alpha \Vdash_0 B$, or

(ii) $\alpha \nVdash_1 A \rightarrow B$.

Note that, w.r.t. the truth condition of implication, we take ($\rightarrow_1$) for L_3, but w.r.t. the falsity condition of implication, we take ($\rightarrow_{0G3}$) and ($\rightarrow_{0RM3}$) for $\mathbf{G}^{wB}_3$ and $\mathbf{IUML}^{-}_3$, respectively. More exactly, the $\mathbf{G}^{wB}_3$-evaluation has the conditions ($-_1$), ($-_0$), ($\wedge_1$), ($\wedge_0$), ($\vee_1$), ($\vee_0$), ($\rightarrow_1$), and ($\rightarrow_{0G3}$); the $\mathbf{IUML}^{-}_3$-evaluation has the conditions (tf_1), (tf_0), ($\top_1$), ($\top_0$), ($\bot_1$), ($\bot_0$), ($-_1$), ($-_0$), ($\sim_1$), ($\sim_0$), ($\wedge_1$), ($\wedge_0$), ($\vee_1$), ($\vee_0$), ($\rightarrow_1$), and ($\rightarrow_{0RM3}$).

A sentence A is *L_3-valid* in a frame $\mathbf{S} = (U, \zeta, \sqsubseteq)$ iff, for all v in $\mathbf{Val}_{L3}$, $\zeta \Vdash^{v}_1 A$. Let Θ be the class of linear frames. A sentence A is L_3-valid, in symbols $\vDash_{L3} A$, iff, for all $\mathbf{S} \in \Theta$, A is *L_3-valid* in $\mathbf{S}$.

Given a class of $\vDash$ $\mathbf{M}_{L3}$ for L_3, we can define (simple truth preserving, corresponding to $\vDash_1$,) consequence as follows:

Definition 3.2 $\Gamma \vDash_{L3} A$ iff, for all $\vDash \mathcal{M} = (U, \zeta, \sqsubseteq, v) \in \mathbf{M}_{L3}$, if $\zeta \Vdash^{v}_1 B$ for all $B \in \Gamma$, then $\zeta \Vdash^{v}_1 A$.}

3.2. Soundness and completeness for L_3

First we note the following lemma, which is useful for the verification of each instance of the axiom schemes in Proposition 3.4 below:

Lemma 3.3 (Hereditary Lemma) For any sentence A,
(i) if $\alpha \Vdash^{v}_{1} A$ and $\alpha \sqsubseteq \beta$, then $\beta \Vdash^{v}_{1} A$, and
(ii) if $\alpha \Vdash^{v}_{0} A$ and $\alpha \sqsubseteq \beta$, then $\beta \Vdash^{v}_{0} A$.

Proof: See Hereditary Lemma in Dunn(1976) and Lemmas 1 and 5 in Yang(2012a). □

Proposition 3.4 (Soundness) If $\vdash_{L_3} A$, then $\vDash_{L_3.} A$.

Proof: The rules of L_3 are (mp) and (adj). Both of these obviously preserve truth, i.e., L_3-validity. (For the former, look at ($\rightarrow_1$) and recall that $\sqsubseteq$ is reflexive; for the latter, look at ($\wedge_1$).) Thus, the proof reduces to verification of axioms for L_3. We verify A18 and A30 as examples.

For A18, we must show that (i) $\alpha \Vdash_1 --A$ only if $\alpha \Vdash_1 A$ and (ii) $\alpha \Vdash_0 A$ only if $\alpha \Vdash_0 --A$. For (i), let $\alpha \Vdash_1 --A$. By ($-_1$) and ($-_0$), we have $\alpha \Vdash_1 --A$ iff $\alpha \Vdash_0 -A$ iff $\alpha \nVdash_0 A$. Then, since the evaluation is total, we obtain $\alpha \Vdash_1 A$. The proof for (ii) is analogous.

For A30, we must show that (i) $\alpha \Vdash_1 (A \rightarrow B) \wedge -(A \rightarrow B)$ only if $\alpha \Vdash_1 --A \wedge -B$ and (ii) $\alpha \Vdash_0 --A \wedge -B$ only if $\alpha \Vdash_0 (A \rightarrow B) \wedge -(A \rightarrow B)$. For (i), let $\alpha \Vdash_1 (A \rightarrow B) \wedge -(A \rightarrow B)$. By

($\wedge_1$), we have $\alpha \Vdash_1 A \rightarrow B$ and $\alpha \Vdash_1 -(A \rightarrow B)$. By ($-_1$) and ($\rightarrow_{0G3}$), we have $\alpha \Vdash_1 -(A \rightarrow B)$ iff $\alpha \Vdash_0 A \rightarrow B$ iff $\alpha \Vdash_0 -A$ and $\alpha \Vdash_0 B$ iff $\alpha \Vdash_1 --A$ and $\alpha \Vdash_1 -B$. Therefore, by ($\wedge_1$), we have $\alpha \Vdash_1 --A \wedge -B$. For (ii), let $\alpha \Vdash_0 --A \wedge -B$. By ($\wedge_0$) and ($-_0$), we have $\alpha \Vdash_0 --A \wedge -B$ iff $\alpha \Vdash_0 --A$ or $\alpha \Vdash_0 -B$ iff $\alpha \nVdash_0 -A$ or $\alpha \nVdash_0 B$. Then, by ($\rightarrow_{0G3}$) and ($-_0$), we further have $\alpha \nVdash_0 -A$ or $\alpha \nVdash_0 B$ iff $\alpha \nVdash_0 A \rightarrow B$ iff $\alpha \Vdash_0 -(A \rightarrow B)$. Then, since $\alpha \Vdash_0 (A \rightarrow B) \wedge -(A \rightarrow B)$ iff $\alpha \Vdash_0 (A \rightarrow B)$ or $\alpha \Vdash_0 -(A \rightarrow B)$ by ($\wedge_0$), we have $\alpha \Vdash_0 (A \rightarrow B) \wedge -(A \rightarrow B)$.

The verification of other axiom schemes for L_3 is left to the reader. □

We give completeness results for L_3 by using the well-known Henkin-style proofs for modal logic, but with prime theories in place of maximal theories. We call a theory Γ *prime* if, for each pair A, B of formulas such that $\Gamma \vdash A \vee B$, $\Gamma \vdash A$ or $\Gamma \vdash B$. By an *L_3-theory*, we mean a theory Γ closed under rules of L_3. As in relevance logic, by a *regular* L_3-theory, we mean an L_3-theory containing all of the theorems of L_3. Since we have no use of irregular theories, from now on, by an L_3-theory, we henceforth mean a regular L_3-theory.

Moreover, where Γ is a prime L_3-theory, we define the *canonical L_3 frame* determined by Γ to be a structure $\mathbf{S} = (U_{can}, \zeta_{can}, \sqsubseteq_{can})$, where ζ_{can} is the Γ, U_{can} is the set of prime L_3 theories extending ζ_{can}, and $\sqsubseteq_{can}$ is $\subseteq$ restricted to U_{can}. Note that the base ζ_{can} is constructed as the prime L_3-theory that excludes nontheorems of L_3, i.e., excludes A such that not $\vdash_{L3}$ A. The partial orderedness and the linear orderedness of the canonical L_3 frame depend on $\subseteq$ restricted on U_{can}. Then, first, the following is obvious.

Proposition 3.5 The canonical L_3 frame is linearly ordered.

Proof: By Proposition 26 in Dunn(2000). □

Next, we define a canonical evaluation as follows:

(1) $1 \in v_{can}(A, \alpha) \Leftrightarrow A \in \alpha$;
(2) $0 \in v_{can}(A, \alpha) \Leftrightarrow -A\ (\sim A \text{ resp}) \in \alpha$.

This definition allows us to state the following lemma.

Lemma 3.6 (Canonical Evaluation Lemma) v_{can} is an evaluation.

Proof: The Hereditary Conditions (HC_1) and (HC_0) are obvious. Thus, we show that the canonical evaluation v_{can} satisfies the truth and falsity conditions above. We prove here the truth and falsity conditions ($-_1$) and ($-_0$) and the falsity condition of implication ($\rightarrow_{0G3}$)

For ($-_1$), we must show

$$\alpha \Vdash^{V_{can}}{}_1 -A \text{ iff } \alpha \Vdash^{V_{can}}{}_0 A.$$

By (1) and (2), we have $\alpha \Vdash^{V_{can}}{}_1 -A$ iff $-A \in \alpha$ iff $\alpha \Vdash^{V_{can}}{}_0 A$.

For ($-_0$), we must show

$$\alpha \Vdash^{Vcan}{}_0 \text{-}A \text{ iff } \alpha \nVdash^{Vcan}{}_0 A.$$

By (2), we have $\alpha \Vdash^{Vcan}{}_0$ -A iff --A $\in \alpha$. Then, since -B for any formula B has boolean properties, we have --B $\in \alpha$ iff -B $\notin \alpha$. Therefore, by (2), we have --A $\in \alpha$ iff -A $\notin \alpha$ iff $\alpha \nVdash^{Vcan}{}_0$ A.

For ($\rightarrow_{0G3}$), we must show

$$\alpha \Vdash^{Vcan}{}_0 A \rightarrow B \text{ iff (i) } \alpha \Vdash^{Vcan}{}_1 \text{--}A \text{ and } \alpha \Vdash^{Vcan}{}_0 B, \text{ or (ii) } \alpha \nVdash^{Vcan}{}_1 A \rightarrow B.$$

For the left-to-right direction, let $\alpha \Vdash^{Vcan}{}_0 A \rightarrow B$. By (1) and (2), we have $\alpha \Vdash^{Vcan}{}_0 A \rightarrow B$ iff -(A $\rightarrow$ B) $\in \alpha$ iff $\alpha \Vdash^{Vcan}{}_1$ -(A $\rightarrow$ B). If A $\rightarrow$ B $\in \alpha$, we obtain --A $\wedge$ -B $\in \alpha$ using A30. Therefore, by (1) and (2), we obtain $\alpha \Vdash^{Vcan}{}_1$ --A and $\alpha \Vdash^{Vcan}{}_0$ B. If A $\rightarrow$ B $\notin \alpha$, we have $\alpha \nVdash^{Vcan}{}_1 A \rightarrow B$. For the right-to-left direction, we first assume $\alpha \Vdash^{Vcan}{}_1$ --A and $\alpha \Vdash^{Vcan}{}_0$ B. Then, using (1), (2), and A31, we can obtain $\alpha \Vdash^{Vcan}{}_0 A \rightarrow B$. Let $\alpha \nVdash^{Vcan}{}_1 A \rightarrow B$. (EM) and primeness ensures $\alpha \Vdash^{Vcan}{}_0 A \rightarrow B$. □

Let us call a model $\mathcal{M}$, = (U, ζ, ⊑, v), for L_3, an L_3 model. Then, by Lemma 3.6, the canonically defined (U_{can}, ζ_{can}, $\sqsubseteq_{can}$, v_{can}) is an L_3 model. Thus, since, by construction, ζ_{can} excludes our chosen nontheorem A, and the canonical definition of ⊨ agrees with membership, we can state that, for each nontheorem A of L_3, there is an L_3 model in which A is not $\zeta_{can} \vDash A$. It gives us the weak completeness of L_3 as follows.

Theorem 3.7 (Weak completeness) If $\vDash_{L3}$ A, then $\vdash_{L3}$ A.

Next, we prove the strong completeness of L_3. As for $\mathbf{R}^+$ in Anderson et al.(1992), we define A to be an *L_3 consequence* of a theory Γ iff for every L_3 model, whenever $\alpha \vDash B$ for every $B \in \Gamma$, $\alpha \vDash A$, for all $\alpha \in U$. We say that A is *L_3 deducible* from Γ iff A is in every L_3-theory containing Γ. Where Δ is a set of formulas not necessarily a theory, $\Delta \vdash A$ can be thought of as saying that A is deducible from the axioms Δ. The set of $\{A: \Delta \vdash A\}$ is intuitively the smallest theory containing the axioms Δ, and we shall label it as *Th(Δ)*. Then,

Proposition 3.8 Let Γ be a theory over L_3. If $\Gamma \nvdash_{L3} A$, then there is a prime theory Γ'' such that $\Gamma \subseteq \Gamma''$ and $A \notin \Gamma''$.}

Proof: We prove the case of $\mathbf{IUML}^-_3$ as an example. Let L_3 be $\mathbf{IUML}^-_3$. Take an enumeration $\{A_n: n \in \omega\}$ of the well-formed formulas of L_3. We define a sequence of sets by induction as follows:

$$\Gamma_0 = \{A': \Gamma \vdash_{L3} A'\}.$$

$$\Gamma_{i+1} = \begin{cases} Th(\Gamma_i \cup \{A_{i+1}\}) & \text{if } \Gamma_i, A_{i+1} \nvdash_{L3} A, \\ \Gamma_i & \text{otherwise.} \end{cases}$$

Let Γ'' be the union of all these Γ_n's. The primeness of Γ'' can be proved using the deduction theorem for $\mathbf{IUML}^-_3$, i.e., Proposition 2.3 (i), along the usual lines. □

Thus, using Lemma 3.6 and Proposition 3.8, we can show strong completeness of L_3 as follows.

Theorem 3.9 (Strong completeness) Let Γ be a theory over L_3. If $\Gamma \models_{L_3} A$, then $\Gamma \vdash_{L_3} A$.

4. Concluding remark

Yang investigated algebraic Kripke-style semantics for three-valued paraconsistent systems in Yang(2014). We further investigated non-algebraic set-theoretical Kripke-style semantics for such systems. But three-valued paraconsistent system having algebraic Kripke-style semantics but not set-theoretical Kripke-style semantics, and vice versa, have not yet been studied. This is a problem left in this paper.

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3치 초일관 논리를 위한 집합-이론적 크립키형 의미론

양 은 석

이 글에서 우리는 3치 초일관 논리를 위한 비대수적 집합-이론적 크립키형 의미론을 다룬다. 이를 위하여 먼저 두 3치 체계를 소개한다. 그리고 그 다음에 이에 상응하는 집합-이론적 크립키형 의미론을 소개한다.

주요어: (집합-이론적) 크립키형 의미론, 대수적 의미론, 3치 논리, 초일관 논리