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An Axiomatic Extension of the Uninorm Logic Revisited*

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【Abstract】In this paper, we show that the standard completeness for the extension of **UL** with compensation-free reinforcement (cfr) $((\phi \,\&\, \psi) \to (\phi \wedge \psi)) \vee ((\phi \vee \psi) \to (\phi \,\&\, \psi))$ can be established. More exactly, first, the compensation-freely reinforced uninorm logic $\mathbf{UL}_{cfr}$ (the **UL** with (cfr)) is introduced. The algebraic structures of $\mathbf{UL}_{cfr}$ are then defined, and its algebraic completeness is established. Next, standard completeness (i.e. completeness on [0, 1]) is established for $\mathbf{UL}_{cfr}$ by using the method introduced in Yang (2009).

【Key Words】(compensation-freely reinforced) fuzzy logic, uninorm, t-norm, algebraic completeness, standard completeness.

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1. Introduction

This paper is a continuation of the work in Yang (2013). First, note that uninorms satisfying t-weakening are t-norms and so the standard completeness proof for the **t**-weakening uninorm logic **UL**$_{\mathrm{Wt}}$, the uninorm logic **UL** with **t**-weakening ($W_{\mathbf{t}}$) $((\phi \,\&\, \psi) \wedge \mathbf{t}) \rightarrow \phi$, introduced in Yang (2009) is not interesting in the sense that such proof is not for a weakening-free uninorm logic. In this paper, we show that such standard completeness can be established for the extension of **UL** with compensation-free reinforcement (cfr) $((\phi \,\&\, \psi) \rightarrow (\phi \wedge \psi)) \vee ((\phi \vee \psi) \rightarrow (\phi \,\&\, \psi))$ as a weakening-free uninorm logic.

We first reconsider the following statements in Yang (2009).

> The starting point for the current work is the observation that t-norms are uninorms. As we mentioned above, while t-norms have unit at 1, uninorms does instead unit lying anywhere in [0, 1]. Then a natural concern arises about for which uninorm logics Metcalfe and Montagna's strategy being able to work. Since **MTL** is the logic of left-continuous t-norms, this strategy of course works for t-norms, i.e., uninorms having identity 1. We here show that it works for other uninorms, i.e., uninorms not being t-norms. More exactly, we show that Jenei and Montagna-style approach may work for logics based on uninorms with a weak form of weakening (called the *t-weakening*), i.e., for *t-weakening uninorm (based) logics*.(Yang (2009), p. 118.)

As the statements show, Yang considered **t**-weakening uninorm logics as logics not based on t-norms. As one example of such uninorm logics, he introduced the **t**-weakening uninorm logic **UL**$_{\mathrm{Wt}}$ and gave standard completeness proof for it in Yang (2009).

However, as Proposition 4.3 in Yang (2013) shows, uninorms satisfying t-weakening are t-norms. The standard completeness for t-norm logics introduced by Jenei and Montagna are well known (see Esteva et al. (2002), Jenei & Montagna (2002)). Thus, this standard completeness proof for $\mathbf{UL}_{\mathrm{Wt}}$ is not interesting since this logic is a t-norm logic, but not a uninorm logic. As a weakening-free logic, here we introduce $\mathbf{UL}_{\mathrm{cfr}}$, the **UL** with compensation-free reinforcement (cfr) $((\phi \& \psi) \to (\phi \wedge \psi)) \vee ((\phi \vee \psi) \to (\phi \& \psi))$, and show that this system is standard complete, i.e., complete with respect to (w.r.t.) the real unit interval [0, 1].

The paper is organized as follows. In Section 2, we present the axiomatization of $\mathbf{UL}_{\mathrm{cfr}}$, which is obtained by adding (cfr) to **UL**. In Section 3, we then define algebraic structures corresponding to the logic $\mathbf{UL}_{\mathrm{cfr}}$, by a subvariety of the variety of commutative residuated lattices (i.e., the variety of $\mathrm{UL}_{\mathrm{cfr}}$-algebras), and show that $\mathbf{UL}_{\mathrm{cfr}}$ is complete w.r.t. linearly ordered $\mathbf{UL}_{\mathrm{cfr}}$-algebras. This will ensure that $\mathbf{UL}_{\mathrm{cfr}}$ is fuzzy in Cintula's sense in Cintula (2006). In Section 4, after defining compensation-freely reinforced uninorms, we note that t-weakening uninorms are t-norms. In Section 5, finally we provide standard completeness results for $\mathbf{UL}_{\mathrm{cfr}}$, using the method introduced in Yang (2009; 2013).[1)]

For convenience, we shall adopt the notation and terminology similar to those in Cintula (2006), Esteva et al. (2002), Hájek

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1) While uninorms have in general the properties of compensation and full reinforcement, t-norms and t-conorms do not. Thus, the standard completeness results show that this method works for full reinforcement, but not for compensation.

(1998), Metcalfe & Montagna (2007), Yang (2009; 2013), and assume familiarity with them (together with the results found therein).

2. Syntax

We base the compensation-freely reinforced fuzzy logic $\mathbf{UL}_{cfr}$ on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives →, &, ∧, ∨, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \to \mathbf{f}$, and
df2. $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$.

We may define **t** as $\mathbf{f} \to \mathbf{f}$. We moreover define $\phi^n{}_{\mathbf{t}}$ as $\phi_{\mathbf{t}}$ & ⋯ & $\phi_{\mathbf{t}}$, n factors, where $\phi_{\mathbf{t}} := \phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of $\mathbf{UL}_{cfr}$ (**UL** plus (cfr)) as a compensation-freely reinforced (substructural) fuzzy logic.

Definition 2.1 $\mathbf{UL}_{cfr}$ consists of the following axiom schemes and rules:

A1. $\phi \to \phi$ (self-implication, SI)
A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ (∧-elimination, ∧-E)

A3. $((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))$ ($\wedge$-introduction, $\wedge$-I)
A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)
A5. $((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi)$ ($\vee$-elimination, $\vee$-E)
A6. $\phi \to \mathbf{T}$ (verum ex quolibet, VE)
A7. $\mathbf{F} \to \phi$ (ex falso quadlibet, EF)
A8. $(\phi \,\&\, \psi) \to (\psi \,\&\, \phi)$ (&-commutativity, &-C)
A9. $(\phi \,\&\, \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)
A10. $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ (suffixing, SF)
A11. $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$ (residuation, RE)
A12. $((\phi \,\&\, \psi) \to (\phi \wedge \psi)) \vee ((\phi \vee \psi) \to (\phi \,\&\, \psi))$
(compensation-free reinforcement, cfr)
A13. $(\phi \to \psi)_{\mathbf{t}} \vee (\psi \to \phi)_{\mathbf{t}}$ ($\mathbf{t}$-prelinearity, $PL_{\mathbf{t}}$).
$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)
$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj)

Proposition 2.2 $\mathbf{UL}_{cfr}$ proves:
(1) $(\phi \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi \,\&\, \psi) \,\&\, \chi)$ (&-associativity, AS).

In $\mathbf{UL}_{cfr}$, $\mathbf{f}$ can be defined as $\sim\mathbf{t}$ and vice versa. A *theory* over $\mathbf{UL}_{cfr}$ is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of $\mathbf{UL}_{cfr}$ or a member of T or follows from some preceding members of the sequence using the rules (mp) and (adj). T $\vdash \phi$, more exactly T $\vdash_{\mathbf{UL}cfr} \phi$, means that ϕ is *provable* in T w.r.t. $\mathbf{UL}_{cfr}$, i.e., there is a $\mathbf{UL}_{cfr}$-proof of ϕ in T. The local deduction theorem (LDT) for $\mathbf{UL}_{cfr}$ is as follows:

Proposition 2.3 Let T be a theory, and ϕ, ψ formulas. $T \cup \{\phi\} \vdash_{\mathbf{ULcfr}} \psi$ iff there is n such that $T \vdash_{\mathbf{ULcfr}} \phi^n_{\mathbf{t}} \to \psi$.

Proof: See Novak (1990). □

A theory T is *inconsistent* if $T \vdash \mathbf{F}$; otherwise it is *consistent*.

For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\to$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

3. Semantics

Suitable algebraic structures for $\mathbf{UL}_{cfr}$ are obtained as a subvariety of the variety of commutative monoidal residuated lattices.

Definition 3.1 A *pointed bounded commutative residuated compensation-freely reinforced lattice* is a structure $\mathbf{A} = (A, \top, \bot, t, f, \wedge, \vee, *, \to)$ such that:

(Ⅰ) $(A, \top, \bot, \wedge, \vee)$ is a bounded lattice with top element $\top$ and bottom element $\bot$.

(Ⅱ) $(A, *, t)$ is a commutative monoid.

(Ⅲ) $y \leq x \to z$ iff $x * y \leq z$, for all $x, y, z \in A$ (residuation).

(Ⅳ) $t \leq ((x*y) \to (x \wedge y)) \vee ((x \vee y) \to (x*y))$, for all $x, y \in A$ (compensation-free reinforcement).

To define the above lattice we may take in place of (III) a family of equations as in Hájek (1998). Using → and f we can define t as f → f, and ∼ as in (df1). In the lattice, ∼ is a *weak* negation in the sense that for all x, $x \leq \sim\sim x$ holds in it. Then, UL_{cfr}-algebras the class of which characterizes $\mathbf{UL_{cfr}}$ are defined as follows.

Definition 3.2 (UL_{cfr}-algebra) A *UL_{cfr}-algebra* is a pointed bounded commutative residuated compensation-freely reinforced lattice satisfying the condition: for all x, y,

(pl_t) $t \leq (x \to y)_t \vee (y \to x)_t$.

A UL_{cfr}-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y. In linearly ordered algebras, we in particular call monoids satisfying (IV) *compensation-freely reinforced monoids*.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function v : FOR → $\mathcal{A}$ satisfying: $v(\phi \to \psi) = v(\phi) \to v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{F}) = \bot$, $v(\mathbf{f}) = f$, (and hence $v(\sim\phi) = \sim v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = t$).

Definition 3.4 Let $\mathcal{A}$ be a UL_{cfr}-algebra, T a theory, φ a formula, and K a class of UL_{cfr}-algebras.

(i) (Tautology) φ is a *t-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or $\mathcal{A}$-*valid*), if v(ϕ) ≥ t for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-*model* of T if v(ϕ) ≥ t for each ϕ ∈ T. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_K \phi$, if Mod(T, $\mathcal{A}$) = Mod(T ∪ {ϕ}, $\mathcal{A}$) for each $\mathcal{A}$ ∈ K.

Definition 3.5 (**UL**$_{cfr}$-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is a ***UL**$_{cfr}$-algebra* iff whenever ϕ is UL$_{cfr}$-provable in T (i.e. $T \vdash_{\mathbf{ULcfr}} \phi$), it is a semantic consequence of T w.r.t. the set {$\mathcal{A}$}, i.e. $T \vDash_{\{A\}} \phi$), $\mathcal{A}$ a UL$_{cfr}$-algebra. By *MOD$^{(l)}$(**UL**$_{cfr}$)*, we denote the class of (linearly ordered) **UL**$_{cfr}$-algebras. Finally, we write $T \vDash^{(l)}_{\mathbf{ULcfr}} \phi$ in place of $T \vDash_{\mathrm{MOD}^{(l)}(\mathbf{ULcfr})} \phi$.

Note that since each condition for the UL$_{cfr}$-algebra has the form of an equation or can be defined in an equation, it can be ensured that the class of all UL$_{cfr}$-algebras is a variety.

Let **A** be a UL$_{cfr}$-algebra. We first show that classes of provably equivalent formulas form a UL$_{cfr}$-algebra. Let T be a fixed theory over **UL**$_{cfr}$. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash_{\mathbf{ULcfr}} \phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \to [\psi]_T = [\phi \to \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \,\&\, \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $\bot = [\mathbf{F}]_T$, $\top = [\mathbf{T}]_T$, $t = [\mathbf{t}]_T$, and $f = [\mathbf{f}]_T$. By $\boldsymbol{A}_T$, we denote this algebra.

Proposition 3.6 For T a theory over L, $\mathbf{A}_T$ is a $\mathbf{UL}_{cfr}$-algebra.

Proof: Note that A1 to A7 ensure that $\wedge$ and $\vee$ satisfy (I) in Definition 3.1; that AS, A8, A9 ensure that & satisfies (II); that A11, A12 and A13 ensure that (III), (IV), and (pl_t) hold. It is obvious that $[\phi]_T \leq [\psi]_T$ iff T $\vdash_{\mathbf{ULcfr}} \phi \leftrightarrow (\phi \wedge \psi)$ iff T $\vdash_{\mathbf{ULcfr}} \phi \rightarrow \psi$. Finally, recall that $\mathbf{A}_T$ is a $\mathbf{UL}_{cfr}$-algebra iff T $\vdash_{\mathrm{ULcfr}} \psi$ implies T $\vDash_{\mathrm{ULcfr}} \psi$, and observe that for ϕ in T, since T $\vdash_{\mathrm{ULcfr}}$ $\mathbf{t} \rightarrow \phi$, it follows that $[\mathbf{t}]_T \leq [\phi]_T$. Thus, it is a $\mathbf{UL}_{cfr}$-algebra. □

We next note that the nomenclature of the prelinearity condition is explained by the subdirect representation theorem below.

Proposition 3.7 Each UL_{cfr}-algebra is a subdirect product of linearly ordered UL_{cfr}-algebras.

Proof: Its proof is analogous to that of Lemma 3.7 in Cintula (2006). □

Theorem 3.8 (Strong completeness) Let T be a theory, and ϕ a formula. T $\vdash_{\mathrm{ULcfr}} \phi$ iff T $\vDash_{\mathrm{ULcfr}} \phi$ iff T $\vDash^{1}_{\mathrm{ULcfr}} \phi$.

Proof: (i) T $\vdash_{\mathrm{ULcfr}} \phi$ iff T $\vDash_{\mathrm{ULcfr}} \phi$. The left-to-right direction follows from definition. The right-to-left direction is as follows: from Proposition 3.6, we obtain $\mathbf{A}_T \in$ MOD(L), and for $\mathbf{A}_T$-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in$ Mod(T, $\mathbf{A}_T$). Thus, since from T $\models_{\mathbf{ULcfr}} \phi$ we obtain that $[\phi]_T = v(\phi) \geq t$, T $\vdash_{\mathbf{ULcfr}}$ $\mathbf{t} \to \phi$. Then, since T $\vdash_{\mathbf{ULcfr}}$ $\mathbf{t}$, by (mp) T $\vdash_{\mathbf{ULcfr}}$ ϕ, as required.

(ii) T $\models_{\mathbf{ULcfr}} \phi$ iff T $\models^{1}_{\mathbf{ULcfr}} \phi$. It follows from Proposition 3.7. □

4. Compensation-freely reinforced uninorms and their residua

In this section, using *1, 0, and some e, and ∂* in the real unit interval [0, 1], we shall express ⊤, ⊥, t, and f, respectively. We also define standard UL_{cfr}-algebras and compensation-freely reinforced uninorms.

Definition 4.1 A UL_{cfr}-algebra is *standard* iff its lattice reduct is [0, 1].

In standard UL_{cfr}-algebras, the monoid operator * is a compensation-freely reinforced uninorm. We first introduce uninorms.

Definition 4.2 A *uninorm* is a function $\bigcirc : [0, 1]^2 \to [0, 1]$ such that for some $e \in [0, 1]$ and for all x, y, z $\in$ [0, 1]:

(a) $x \bigcirc y = y \bigcirc x$ (commutativity),
(b) $x \bigcirc (y \bigcirc z) = (x \bigcirc y) \bigcirc z$ (associativity),

(c) $x \leq y$ implies $x \circ z \leq y \circ z$ (monotonicity), and
(d) $e \circ x = x$ (identity).

Uninorms satisfying (1-identity) $e = 1$ are *t-norms*. $\circ$ is *residuated* iff there is $\rightarrow : [0, 1]^2 \rightarrow [0, 1]$ satisfying (residuation) on [0, 1]. A uninorm is called *conjunctive* if $0 \circ 1 = 0$, and *disjunctive* if $0 \circ 1 = 1$. For some $\partial \in [0, 1]$, a residuated uninorm has weak negation n defined as $n(x) := x \rightarrow \partial$ because $x \circ (x \rightarrow \partial) \leq \partial$ holds in it and so by residuation $x \circ (x \rightarrow \partial) \leq \partial$ iff $x \leq (x \rightarrow \partial) \rightarrow \partial$.

The most important property of a uninorm is that *left-continuity* holds in it. Given a uninorm $\circ$, *residuated implication* $\rightarrow$ determined by $\circ$ is defined as $x \rightarrow y := \sup\{z \in [0, 1]: x \circ z \leq y\}$ for all $x, y \in [0, 1]$. Then, we can show that for any uninorm $\circ$, $\circ$ and its residuated implication $\rightarrow$ form a residuated pair iff $\circ$ is conjunctive and left-continuous in both arguments (see Proposition 5.4.2 in Gottwald (2001)).

A compensation-freely reinforced uninorm is defined as follows.

Definition 4.3 A *compensation-freely reinforced* uninorm is a residuated uninorm satisfying for all $x, y \in [0, 1]$:

(cfr) $x \circ y \leq \min\{x, y\}$ or $\max\{x, y\} \leq x \circ y$.

Notice that (cfr) ensures that compensation-freely reinforced uninorms can be defined as residuated uninorms satisfying: for all $x, y \in [0, 1]$, (cfr$'$) $e \leq \max\{(x \circ y) \rightarrow \min(x, y), \max(x,$

$y) \rightarrow (x \bigcirc y)\}$.

Example 4.4 (i) (Yang (2011)) Given a *fixed-point weak* negation n, i.e., a negation n satisfying: for all $x \in [0, 1]$,

(a) $n(t) = t$,

(b) $n(n(x)) \geq x$, and

(c) $n(0) = 1$ and $n(1) = 0$,

we can construct a conjunctive left-continuous idempotent uninorm $\bigcirc$ given by, for all $x, y \in [0, 1]$:

$$x \bigcirc y = \begin{cases} \min(x, y) & \text{if } y \leq n(x), \\ \max(x, y) & \text{otherwise.} \end{cases}$$

(ii) (Klement et al. (2000)) Given the *standard negation* $n_s = 1 - x$, we can construct a conjunctive left-continuous idempotent uninorm $\bigcirc_s$ given by, for all $x, y \in [0, 1]$:

$$x \bigcirc_s y = \begin{cases} \min(x, y) & \text{if } x + y \leq 1, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

(iii) (Klement et al. (2000)) Consider a conjunctive left-continuous idempotent uninorm $\bigcirc$ with a negation n. Then its residuated implication $\rightarrow$ is given by

$$x \rightarrow y = \begin{cases} \max(n(x), y) & \text{if } x \leq y, \\ \min(n(x), y) & \text{otherwise.} \end{cases}$$

(iv) (De Baets & Fodor (1999), Klement et al. (2000)) Consider the standard negation n_s. Then the residiated implication $\rightarrow_s$ of the corresponding conjunctive left-continuous idempotent uninorm $\bigcirc_s$ is given by

$$x \rightarrow_s y = \begin{cases} \max(1\text{-}x, y) & \text{if } x \leq y, \\ \min(1\text{-}x, y) & \text{otherwise.} \end{cases}$$

The structure $\mathbf{A}_s$ = ([0, 1], 1, 0, ½, ½, min, max, $\bigcirc_s$, $\to_s$), where $\bigcirc_s$ and $\to_s$ are the conjunctive left-continuous idempotent uninorm and its residuum, is known to us as the algebra for the involutive uninorm mingle logic **IUML**.

Fact 4.5 (Metcalfe & Montagna (2007)) Let $\mathbf{A}_s$ = ([0, 1], 1, 0, ½, ½, min, max, $\bigcirc_s$, $\to_s$), where:

$$x \bigcirc_s y = \min(x, y) \text{ if } x + y \leq 1,$$
$$\max(x, y) \text{ otherwise.}$$

ϕ is valid in all standard IUML-algebras iff ϕ is valid in the IUML-algebra $\mathbf{A}_s$.

Note that the conjunctive left-continuous idempotent uninorm $\bigcirc_s$ does not satisfy (1-identity), and so not forms t-norms.

Note 4.6 In Yang (2013), Yang verified that uninorms satisfying (t-weakening) are t-norms. We remind this: Given any t-weakening uninorm $\bigcirc$, for all $x < e$, we have $\min\{x \bigcirc 1, e\} \leq x$, and hence, $x \bigcirc 1 \leq x$. Since $x = x \bigcirc e \leq x \bigcirc 1$, for all $x < e$, we have $x \bigcirc 1 = x$. By the left-continuity of $\bigcirc$, $e \bigcirc 1 = \sup\{x \bigcirc 1: x < e\} = \sup\{x: x < e\} = e$. But since $e \bigcirc 1 = 1$, $1 = e$, and the uninorm is a t-norm.

5. Standard completeness

We first show that finite or countable linearly ordered

$\mathbf{UL}_{cfr}$-algebras are embeddable into a standard algebra. (For convenience, we add less than relation symbol to such algebras.)

Proposition 5.1 For every finite or countable linearly ordered $\mathbf{UL}_{cfr}$-algebra $\mathbf{A} = (A, \leq_A, \top, \bot, t, f, \wedge, \vee, *, \rightarrow)$, there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the following conditions hold:

(Ⅰ) X is densely ordered, and has a maximum Max, a minimum Min, and special elements e, ∂.

(Ⅱ) $(X, \circ, \preceq, e)$ is a linearly ordered monotonic commutative compensation-freely reinforced monoid.

(Ⅲ) $\circ$ is conjunctive and left-continuous w.r.t. the order topology on $(X, \preceq)$.

(Ⅳ) h is an embedding of the structure $(A, \leq_A, \top, \bot, t, f, \wedge, \vee, *)$ into $(X, \preceq, Max, Min, e, \partial, min, max, \circ)$, and for all $m, n \in A$, $h(m \rightarrow n)$ is the residuum of $h(m)$ and $h(n)$ in $(X, \preceq, Max, Min, e, \partial, max, min, \circ)$.

Proof: For convenience, we assume A as a subset of $\mathbf{Q} \cap [0, 1]$ with finite or countable elements, where 0 and 1 are least and greatest elements, each of which corresponds to $\top$, $\bot$, respectively. Let

$$X = \{(m, x): m \in A \setminus \{0\ (= \bot)\} \text{ and } x \in \mathbf{Q} \cap (0, m]\} \cup \{(0, 0)\}.$$

For $(m, x), (n, y) \in X$, we define:

$$(m, x) \leq (n, y) \text{ iff either } m <_A n, \text{ or } m =_A n \text{ and } x \leq y.$$

It is clear that $\leq$ is a linear order with maximum (1, 1), minimum (0, 0), and special elements e = (t, t), ∂ = (f, f). Furthermore, $\leq$ is dense: let (m, x) $<$ (n, y). Then either $m <_A n$ or $m =_A n$ and $x < y$. If the first is the case, then (m, x) $<$ (n, y/2) $<$ (n, y). Otherwise, (m, x) $<$ (n, x+y/2) $<$ (n, y). This proves (I).

For convenience, we will henceforth drop the index A in $<_A$ and $=_A$, if we need not distinguish them. But context should make clear what we mean.

Define for (m, x), (n, y) $\in$ X:

$$(m,x) \circ (n,y) = \begin{cases} \max\{(m,x), (n,y)\} & \text{if } m * n = m \vee n, \text{ and } (m, x) > e \text{ or } (n, y) > e; \\ \min\{(m,x), (n,y)\} & \text{if } m * n = m \wedge z, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \\ (m * n, m * n) & \text{otherwise.} \end{cases}$$

We verify that $\circ$ satisfies (II) (noting that (cfr) of * ensures that for all m, n $\in$ A, $m * n \leq m \wedge n$ or $m \vee n \leq m * n$).

(1) Commutativity. It is obvious that $\circ$ is commutative.

(2) Identity. We prove that (t, t) is the unit element, i.e., (t, t) $\circ$ (m, x) = (m, x). (i) Let (t, t) $<$ (m, x). Since $t * m = m = t \vee m$, (t, t) $\circ$ (m, x) = max{(t, t), (m, x)} = (m, x). (ii) Let (m, x) $\leq$ (t, t). Since $t * m = m = t \wedge m$, (t, t) $\circ$ (m, x) = min{(t, t), (m, x)} = (m, x).

(3) Monotonicity. Since $\circ$ is commutative, it suffices to prove that if $(l, x) \leq (m, y)$, then for all $(n, z) \in X$, $(l, x) \circ (n, z) \leq (m, y) \circ (n, z)$. We distinguish several cases:

- Case (i). $l * n = l \vee n$ and $m * n = m \vee n$:

Subcase (i-a). $(l, x) > e$ or $(n, z) > e$.

(a-1) $(m, y) > e$ or $(n, z) > e$. If $e < (l, x), (n, z), (m, y)$, then $(l, x) \circ (n, z) = \max\{(l, x), (n, z)\} \leq \max\{(m, y), (n, z)\} = (m, y) \circ (n, z)$. If $(n, z) \leq e < (l, x) \leq (m, y)$, $(l, x) \circ (n, z) = \max\{(l, x), (n, z)\} = (l, x) \leq (m, y) = \max\{(m, y), (n, z)\} = (m, y) \circ (n, z)$. If $(l, x) \leq e < (n, z)$, $(l, x) \circ (n, z) = \max\{(l, x), (n, z)\} = (n, z) \leq \max\{(m, y), (n, z)\} = (m, y) \circ (n, z)$.

(a-2) $(m, y), (n, z) \leq e$. This is not the case by the supposition.

Subcase (i-b). $(l, x), (n, z) \leq e$.

(b-1) $(m, y) > e$. Then $(l, x) \circ (n, z) = \min\{(l, x), (n, z)\} < (m, y) = \max\{(m, y), (n, z)\} = (m, y) \circ (n, z)$.

(b-2) $(m, y) \leq e$. Then $l = m = n$, and so $(l, x) \circ (n, z) = \min\{(l, x), (n, z)\} \leq \min\{(m, y), (n, z)\} = (m, y) \circ (n, z)$.

- Case (ii). $l * n = l \wedge n$ and $m * n = m \wedge n$. Its proof is analogous to that of Case (i).

- Case (iii). $l * n = l \vee n$ and $m * n \neq m \vee n$. We need to consider the subcases (a) $m * n = m \wedge n$ and (b) $m *$

$n \neq m \wedge n$.

Subcase (iii-a). $m * n = m \wedge n$. Since $m * n \neq m \vee n$ and so $m \neq n$, $l = n < m, t$. Then $(l, x) \circ (n, z) = \min\{(l, x), (n, z)\} \leq \min\{(m, y), (n, z)\} = (m, y) \circ (n, z)$.

Subcase (iii-b). $m * n \neq m \wedge n$:

(b-1) $m * n > t$. Then, since $l * n \leq m * n$ and $(m, y) \circ (n, z) = (m * n, m * n)$, $(l, x) \circ (n, z) \leq (m, y) \circ (n, z)$.

(b-2) $m * n \leq t$. This is not the case because it implies $l = n = l * n < m * n \leq t$, but $m * n \leq m \wedge n$ or $m \vee n \leq m * n$ and so $m * n < m \wedge n = n$.

- Case (iv). $l * n \neq l \vee n$ and $m * n = m \vee n$. Its proof is analogous to that of Case (iii).

- Case (v). $l * n \neq l \vee n, l \wedge n$, and $m * n \neq m \vee n, m \wedge n$.

Subcase (v-a). $l * n, m * n > t$. $(l, x) \circ (n, z) = (l * n, l * n) \leq (m * n, m * n) = (m, y) \circ (n, z)$.

Subcase (v-b). $l * n \leq t < m * n$. If $n \leq t$, then $m * n \leq m \vee n$, and otherwise, $l \wedge n \leq l * n$. Thus, this is not the case.

Subcase (v-c). $l * n > t \geq m * n$. By the supposition, this is not the case.

Subcase (v-d). Otherwise, i.e., $l * n, m * n \leq t$. $(l, x) \circ (n, z) = (l * n, l * n) \leq (m * n, m * n) = (m, y) \circ (n, z)$.

(4) Compensation-free reinforcement. (i) Let $m * n \leq m \wedge n$. If

$m * n = m \wedge n \leq t$, then $(m, x) \circ (n, y) = \min\{(m, x), (n, y)\}$. If $m * n = m \wedge n > t$, then, $(m, x) \circ (n, y) = \max\{(m, x), (n, y)\}$. Otherwise, i.e., if $m * n < m \wedge n$, then $(m, x) \circ (n, y) \prec \min\{(m, x), (n, y)\}$. (ii) Let $m \vee n \leq m * n$. Its proof is analogous to that of (i).

(5) Associativity. We show that for all $(l, x), (m, y), (n, z) \in X$,

$$(l, x) \circ ((m, y) \circ (n, z)) = ((l, x) \circ (m, y)) \circ (n, z) \text{ (AS)}.$$

Without further mention, we will use the fact that * is associative and compensation-freely reinforced. We distinguish several cases:

• Case (i). $l * (m * n) = \vee(l, m, n)$. If either $t < l$ and $t < l * (m * n)$, or $t < m$ and $t < l * (m * n)$, or $t < n$ and $t < l * (m * n)$, then both sides of (AS) are equal to $\max\{(l, x), (m, y), (n, z)\}$. Otherwise, both sides of (AS) are equal to $\min\{(l, x), (m, y), (n, z)\}$.

• Case (ii). $l * (m * n) = \wedge(l, m, n)$. If $t < l = m = n$, both sides of (AS) are equal to $\max\{(l, x), (m, y), (n, z)\}$. Otherwise, both sides of (AS) are equal to $\min\{(l, x), (m, y), (n, z)\}$.

• Case (iii). $l * (m * n) \neq \vee(l, m, n), \wedge(l, m, n)$, and $l * (m * n) \in \{l, m, n\}$. This is not the case because $\vee(l, m, n) \leq l * (m * n)$ or $l * (m * n) \leq \wedge(l, m, n)$ by (cfr).

• Case (iv). $l * (m * n) \notin \{l, m, n\}$ and either $l * (m * n) = l \vee (m * n) = m * n$ or $l * (m * n) = l \wedge (m * n) = m * n$. Then, since (cfr) ensures that either $t \leq l, m \vee n < m * n$ or $t \geq l, m \wedge n > m * n$, both sides of (AS) are equal to $(m * n, m * n)$.

• Case (v). $l * (m * n) \notin \{l, m, n\}$ and $l * (m * n) \neq l \vee (m * n), l \wedge (m * n)$. Then, we need to consider the cases $l * (m * n) > t$ and $l * (m * n) \leq t$. Both sides of (AS) are equal to $(l * (m * n), l * (m * n))$.

We then prove (Ⅲ). Since $0 * 1 = 0$, it is immediate that ○ is conjunctive, i.e., $(0, 0) \circ (1, 1) = (0, 0)$.

For left-continuity of ○, we prove that if $\langle (m_i, x_i): i \in \mathbf{N} \rangle$ is any increasing sequence (w.r.t. $\leq$) of elements of X such that $\sup\{(m_i, x_i): i \in \mathbf{N}\} = (m, x)$, then for all $(n, y) \in X$, $\sup\{(m_i, x_i) \circ (n, y): i \in \mathbf{N}\} = (m, x) \circ (n, y)$. Note that for almost all i, $m_i = m$ (otherwise $(m, x/2) \prec (m, x)$ would be an upper bound of the sequence $\langle (m_i, x_i): i \in \mathbf{N} \rangle$). By deleting a finite number of elements of the sequence $\langle (m_i, x_i): i \in \mathbf{N} \rangle$, we can suppose that for all i, $m_i = m$ and that $x = \sup\{x_i: i \in \mathbf{N}\}$. Then we need to consider the following cases:

Case (i). $m * n = m \vee n$. In case $m > t$ or $n > t$, $(m, x) \circ (n, y) = \max\{(m, x), (n, y)\}$, $(m_i, x_i) \circ (n, y) = \max\{(m_i, x_i), (n, y)\}$, and left-continuity follows from left-continuity of max operation. Otherwise, i.e., if $m = n \leq t$, $(m, x) \circ (n, y) = \min\{(m, x), (n, y)\}$ and for all i, $(m_i, x_i) \circ (n, y) = (\min\{(m_i,$

x), (n, y)}), and left-continuity follows from left-continuity of min operation.

Case (ii). $m * n = m \wedge n$. Its proof is analogous to that of Case (i).

Case (iii). $m * n \neq m \vee n, m \wedge n$. Then, $(m, x) \circ (n, y) = (m * n, m * n)$ and for all i, $(m_i, x_i) \circ (n, y) = (m_i * n, m_i * n) = (m * n, m * n)$. Thus $(m, x) \circ (n, y) = (m_i, x_i) \circ (n, y)$.

This completes the proof of (Ⅲ).

We finally prove (Ⅳ). First define for every $m \in A$,

$$h(m) = (m, m).$$

It is clear that h is increasing and so one-to-one. h(1), h(0), h(t), and h(f) are top, bottom, and special elements of $(X, \leq)$; and h(t) is the unit element of $\circ$. We then show that $h(m) \circ h(n) = h(m * n)$:

Case (i). $t < m, n$. $h(m) \circ h(n) = (m, m) \circ (n, n) = (m * n, m * n) = h(m * n)$.

Case (ii). $m \leq t < n$.

Subcase (ii-a). $m * n = m \vee n$. $h(m) \circ h(n) = (m, m) \circ (n, n) = \max\{(m, m), (n, n)\} = (n, n) = h(n) = h(m * n)$.

Subcase (ii-b). $m * n = m \wedge n$. $h(m) \circ h(n) = (m, m) \circ (n, n) = \min\{(m, m), (n, n)\} = (m, m) = h(m) = h(m * n)$.

Case (iii). $n \leq t < m$. Its proof is analogous to that of Case (ii).

Case (iv). $t \geq m, n$. Its proof is analogous to that of Case (i). Thus h is an embedding of partially ordered monoids. It remains to prove that for every $l, m, n \in A$, $h(l \to m)$ is the residuum of h(l) and h(m) w.r.t. ○, i.e., (i) h(l) ○ h(l → m) ≤ h(m), and (ii) if h(l) ○ (n, z) ≤ h(m), then (n, z) ≤ h(l → m).

(i). Consider the case t < 1 ≤ m. h(l) ○ h(l → m) = (l, 1) ○ (l → m, l → m) = (l * (l → m), l * (l → m)) ≤ (m, m) = h(m). Proof of the other cases is analogous.

(ii). By contraposition, we prove this. Suppose that h(l → m) ≺ (n, z), i.e., (l → m, l → m) ≺ (n, z). Since l → m is the residuum of l and m in A, m < l * n. Thus (m, m) ≺ (l, l) ○ (n, z). This completes the proof. □

Proposition 5.2 Every countable linearly ordered $\mathbf{UL_{cfr}}$-algebra can be embedded into a standard algebra.

Proof: In an analogy to the proof of Theorem 3.2 in Jenei & Montagna (2002), we prove this. Let X, **A**, etc. be as in Proposition 5.1. Since (X, ≤) is a countable, dense, linearly-ordered set with maximum and minimum, it is order isomorphic to (**Q** ∩ [0, 1], ≤). Let g be such an isomorphism. If (I), (II), (III), and (IV) hold, letting for α, β ∈ [0, 1], α ○′ β = $g(g^{-1}(\alpha) \circ g^{-1}(\beta))$, and, for all m ∈ A, h′(m) = g(h(m)), we obtain that **Q** ∩ [0, 1], ≤, 1, 0, e, ∂, ○′, h′ satisfy the conditions (I) to (IV) of Proposition 5.1 whenever X, ≤, Max, Min, e, ∂, ○, and h do. Thus we can without loss of generality assume that X = **Q** ∩ [0, 1] and ≤ = ≤.

Now we define for α, β ∈ [0, 1],

$$\alpha \circ'' \beta = \sup_{x \in X: x \leq \alpha} \sup_{y \in X: y \leq \beta} x \circ y.$$

Commutativity of $\bigcirc''$ follows from that of $\bigcirc$. Its monotonicity, identity, and compensation-free reinforcement are easy consequences of the definition. Furthermore, it follows from the definition that $\bigcirc''$ is conjunctive, i.e., $0 \bigcirc'' 1 = 0$.

We prove left-continuity. Suppose that $<\alpha_n: n \in \mathbf{N}>$, $<\beta_n: n \in \mathbf{N}>$ are increasing sequences of reals in [0, 1] such that $\sup\{\alpha_n: n \in \mathbf{N}\} = \alpha$ and $\sup\{\beta_n: n \in \mathbf{N}\} = \beta$. By the monotonicity of $\bigcirc''$, $\sup\{\alpha_n \bigcirc'' \beta_n\} = \alpha \bigcirc'' \beta$. Since the restriction of $\bigcirc''$ to $\mathbf{Q} \cap [0, 1]$ is left-continuous, we obtain

$$\begin{aligned} \alpha \bigcirc'' \beta &= \sup\{q \bigcirc'' r: q, r \in \mathbf{Q} \cap [0, 1], q \leq \alpha, r \leq \beta\} \\ &= \sup\{q \bigcirc'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\}. \end{aligned}$$

For each $q < \alpha$, $r < \beta$, there is n such that $q < \alpha_n$ and $r < \beta_n$. Thus,

$$\begin{aligned} \sup\{\alpha_n \bigcirc'' \beta_n: n \in \mathbf{N}\} \geq \sup\{q \bigcirc'' r: q, r \in \mathbf{Q} \cap [0, 1], \\ q < \alpha, r < \beta\} = \alpha \bigcirc'' \beta. \end{aligned}$$

Hence, $\bigcirc''$ is a left-continuous compensation-freely reinforced uninorm on [0, 1].

It is an easy consequence of the definition that $\bigcirc''$ extends $\bigcirc$. By (Ⅰ) to (Ⅳ), h is an embedding of $(A, \leq_A, \top, \bot, t, f, \wedge, \vee, *)$ into $([0, 1], \leq, 1, 0, e, \partial, \min, \max, \bigcirc'')$. Moreover, $\bigcirc''$ has a residuum, calling it $\rightharpoonup$.

We finally prove that for x, y $\in$ A, $h(x \rightarrow y) = h(x) \rightharpoonup h(y)$.

By (Ⅳ), $h(x \rightarrow y)$ is the residuum of $h(x)$ and $h(y)$ in $(\mathbf{Q} \cap [0, 1], \leq, 1, 0, e, \partial, \min, \max, \circ)$. Thus

$$h(x) \circ'' h(x \rightarrow y) = h(x) \circ h(x \rightarrow y) \leq h(y).$$

Suppose toward contradiction that there is $\alpha > h(x \rightarrow y)$ such that $\alpha \circ'' h(x) \leq h(y)$. Since $\mathbf{Q} \cap [0, 1]$ is dense in $[0, 1]$, there is $q \in \mathbf{Q} \cap [0, 1]$ such that $h(x \rightarrow y) < q \leq \alpha$. Hence $q \circ'' h(x) = q \circ h(x) \leq h(y)$, contradicting (Ⅳ). □

Theorem 5.3 (Strong standard completeness) For $\mathbf{UL}_{cfr}$, the following are equivalent:

(1) $T \vdash_{\mathbf{ULcfr}} \phi$.

(2) For every standard $\mathbf{UL}_{cfr}$-algebra and evaluation v, if $v(\psi) \geq e$ for all $\psi \in T$, then $v(\phi) \geq e$.

Proof: (1) to (2) follows from definition. We prove (2) to (1). Let ϕ be a formula such that $T \nvdash_{\mathbf{ULcfr}} \phi$, **A** a linearly ordered $\mathbf{UL}_{cfr}$-algebra, and v an evaluation in **A** such that $v(\psi) \geq t$ for all $\psi \in T$ and $v(\phi) < t$. Let h' be the embedding of **A** into the standard $\mathbf{UL}_{cfr}$-algebra as in proof of Proposition 5.2. Then $h' \circ v$ is an evaluation into the standard $\mathbf{UL}_{cfr}$-algebra such that $h' \circ v(\psi) \geq e$ and yet $h' \circ v(\phi) < e$. □

Theorem 5.3 ensures that $\mathbf{UL}_{cfr}$ is complete w.r.t. left-continuous conjunctive compensation-freely reinforced uninorms and their residua, i.e., for each formula ϕ, if $\nvdash_{\mathbf{ULcfr}} \phi$, then there is a left-continuous conjunctive compensation-freely reinforced uninorm ○ and an evaluation v into ([0, 1], ○″, →, ≤, 1, 0, e, ∂), where → is the residuum of ○″, such that v(ϕ) < e.

6. Concluding remark

We investigated (not merely algebraic completeness but also) standard completeness for $\mathbf{UL}_{cfr}$. This work can be generalized to the systems, which are axiomatic extensions of $\mathbf{UL}_{cfr}$. We shall investigate this in some subsequent paper.

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유니놈 논리의 확장을 재고함

양 은 석

이 글에서 우리는 보상 없는 강화 (cfr) ((φ & ψ) → (φ ∧ ψ)) ∨ ((φ ∨ ψ) → (φ & ψ))를 갖는 유니놈 논리의 확장에 대해 표준 완전성이 제공될 수 있다는 것을 보인다. 이를 위하여, 먼저 보상 없는 강화를 갖는 유니놈 논리 $\mathbf{UL}_{cfr}$을 소개한다. 이 체계에 상응하는 대수적 구조를 정의한 후, $\mathbf{UL}_{cfr}$이 대수적으로 완전하다는 것을 보인다. 다음으로, $\mathbf{UL}_{cfr}$이 표준적으로 완전하다는 것 즉 단위 실수 [0, 1]에서 완전하다는 것을 Yang (2009)에서의 방법을 사용하여 보인다.

주요어: (보상 없는 강화) 퍼지 논리, 유니놈, t-규범, 대수적 완전성, 표준 완전성