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Algebraic Kripke-style semantics for weakening-free fuzzy logics*

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【Abstract】This paper deals with Kripke-style semantics for fuzzy logics. More exactly, I introduce algebraic Kripke-style semantics for some weakening-free extensions of the uninorm based fuzzy logic **UL**. For this, first, I introduce several weakening-free extensions of **UL**, define their corresponding algebraic structures, and give algebraic completeness. Next, I introduce several algebraic Kripke-style semantics for those systems, and connect these semantics with algebraic semantics.

【Key Words】Kripke-style semantics, Algebraic semantics, Many-valued logic, Fuzzy logic

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1. Introduction

After introducing algebraic semantics for t-norm and uninorm (based) logics, their corresponding algebraic Kripke-style semantics have been introduced. For instance, after Esteva and Godo introducing algebraic semantics for monoidal t-norm (based) logics in Esteva & Godo (2001), their corresponding algebraic Kripke-style semantics were introduced in Montagna & Ono (2002) and Montagna & Sacchetti (2003; 2004). Analogously, after Metcalfe and Montagna introducing algebraic semantics for the uninorm (based) logic **UL** in Metcalfe & Montagna (2007), its corresponding algebraic Kripke-style semantics was introduced in Yang (2012).

This paper is a continuation of the work in Yang (2012). Note that, although Metcalfe and Montagna introduced algebraic semantics for some weakening-free extensions of **UL**, Kripke-style semantics for these logics have not yet been introduced. Note also that in Yang (2012), he said that "we did not provide algebraic Kripke-style semantics for axiomatic extensions of **UL**. We will investigate it in a subsequent paper(Yang (2012), p. 13)." By providing algebraic Kripke-style semantics for some weakening-free extensions of **UL**, this paper completes Yang's idea.

For this, first, in Section 2, we recall **UL** and its weakening-free extensions introduced in Metcalfe & Montagna (2007), and their corresponding algebraic semantics as the necessary notions for treating the question in Yang (2012). In

Section 3, we introduce algebraic Kripke-style semantics for such extensions, and connect them with algebraic semantics.

Note that many logicians have introduced algebraic semantics as semantics for fuzzy logic, whereas other logicians have disliked such semantics due to a lack of philosophical implication. In particular, some logicians complained that fuzzy logic does not have semantics such as world semantics for modal logic. As is known, Kripke semantics is a representative of such semantics. This investigation will show that fuzzy logic also have such semantics.

For convenience, we shall adopt notation and terminology similar to those in Cintula et al (2009), Metcalfe & Montagna (2007), Montagna & Sacchetti (2003; 2004), and Yang (2012), and we assume reader's familiarity with them (along with results found therein).

2. Weakening-free uninorm logics and their algebraic semantics

We base **UL** and its weakening-free extensions on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$, and

df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We moreover define ϕ^n_t as ϕ_t & ⋯ & ϕ_t, n factors, where ϕ_t := $\phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use axiom systems to provide a consequence relation.

We start with the following axiomatization of **UL** as the most basic (substructural) fuzzy logic introduced here.

Definition 2.1 (Metcalfe & Montagna (2007)) **UL** consists of the following axiom schemes and rules:

A1. $\phi \to \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\phi\to\psi)\wedge(\phi\to\chi)) \to (\phi\to(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi\to\chi)\wedge(\psi\to\chi)) \to ((\phi\vee\psi)\to\chi)$ ($\vee$-elimination, $\vee$-E)

A6. $\phi \to \mathbf{T}$ (verum ex quolibet, VE)

A7. $\mathbf{F} \to \phi$ (ex falso quadlibet, EF)

A8. $(\phi \,\&\, \psi) \to (\psi \,\&\, \phi)$ (&-commutativity, &-C)

A9. $(\phi \,\&\, \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)

A10. $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ (suffixing, SF)

A11. $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$ (residuation, RE)

A12. $(\phi \to \psi)_t \vee (\psi \to \phi)_t$ (**t**-prelinearity, PL_t)

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj).

Definition 2.2 (ULs) A logic is a schematic extension of **L** if and only if (iff) it results from **L** by adding axiom schemes. **L** is a UL iff **L** is a schematic extension of **UL**. In particular, the following are weakening-free extensions introduced in Metcalfe & Montagna (2007).

- **IUL** is **UL** plus $\sim\sim\phi \rightarrow \phi$ (double negation elimination, DNE)
- **UML** is **UL** plus $(\phi \,\&\, \phi) \leftrightarrow \phi$ (idempotence, ID)
- **IUML** is **IUL** plus (ID) and $\mathbf{t} \leftrightarrow \mathbf{f}$ (fixed-point, FP)

For easy reference we group the weakening-free uninorm logics introduced in Definitions 2.1 and 2.2 as a set.

Definition 2.3 *Ls* = {**UL**, **IUL**, **UML**, **IUML**}

A *theory* over L ($\in$ Ls) is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using the two rules in Definition 2.1. $T \vdash \phi$, more exactly $T \vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. The (local) deduction theorem ((L)DT_t) for L is as follows:

Proposition 2.4 Let T be a theory over L, and ϕ, ψ formulas.
(i) (LDT_t) $T \cup \{\phi\} \vdash \psi$ iff there is n such that $T \vdash \phi^n_{\mathbf{t}} \rightarrow \psi$.
(ii) (DT_t) For L with ID, $T \cup \{\phi\} \vdash \psi$ iff $T \vdash \phi_{\mathbf{t}} \rightarrow \psi$.

A theory T is *inconsistent* if $T \vdash \mathbf{F}$; otherwise it is *consistent*.

For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

The algebraic counterpart of L is the class of the so-called *L-algebras*. Let $x_t := x \wedge t$. They are defined as follows.

Definition 2.5 (Metcalfe & Montagna (2007)) Define a relation $\leq$ so that $x \leq y$ iff $x \wedge y = x$.

(i) (UL-algebra) A *UL algebra* is a structure **A** = (A, ⊤, ⊥, t, f, ∧, ∨, *, →) such that:

(I) (A, ⊤, ⊥, ∧, ∨) is a bounded lattice with top element ⊤ and bottom element ⊥.

(II) (A, *, t) is a commutative monoid.

(III) $y \leq x \rightarrow z$ iff $x * y \leq z$, for all $x, y, z \in A$ (residuation).

(IV) $t \leq (x \rightarrow y)_t \vee (y \rightarrow x)_t$ (pl_t).

(ii) The other algebras are defined as follows: An *IUL*-algebra is a UL-algebra satisfying (dne) $(x \rightarrow f) \rightarrow f \leq x$. A *UML*-algebra is a UL-algebra satisfying (id) $x = x * x$. An *IUML*-algebra is an IUL-algebra satisfying (id) and (fp) $t = f$.

Additional (unary) negation and (binary) equivalence operations are defined as follows: $\sim x := x \rightarrow f$ and $x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x)$.

The class of all L-algebras is a variety which will be denoted by $\mathbb{L}$.

L-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 2.6 Let K be a class of L-algebras. We define consequence relation $\vDash_{\mathsf{K}}$ in the following way: T $\vDash_{\mathsf{K}}$ ϕ iff for each $\mathbf{A} \in \mathsf{K}$ and $\mathbf{A}$-evaluation v, we have v(A) $\geq$ t whenever v(ψ) $\geq$ t for each $\psi \in$ T. If K be a class of linearly ordered L-algebras, we denote consequence relation as $\vDash^{l}_{L}$.

We write $\vDash_{\mathsf{K}} \phi$ instead of $\varnothing \vDash_{\mathsf{K}} \phi$, and T $\vDash_{\mathbf{A}} \phi$ instead of T $\vDash_{\{A\}} \phi$.

That L is the proper algebraic semantics for L is witnessed by the following completeness result.

Theorem 2.7 (Metcalfe & Montagna (2007)) Let T be a theory over L ($\in$ Ls), and ϕ a formula. T $\vdash_{L} \phi$ iff T $\vDash_{\mathsf{L}} \phi$.

This completeness result can be refined by taking into account the following representation of L-algebras related to the prelinearity property of L-algebras.

Proposition 2.8 (Tsinakis & Blount (2003)) Each L-algebra is a subdirect product of linearly ordered L-algebras.

Linearly ordered sets are also called chains. This proposition leads to the completeness of L w.r.t. the class of chains of L.

Corollary 2.9 (Metcalfe & Montagna (2007)) Let T be a theory over L, and ϕ a formula. T $\vdash_{L} \phi$ iff T $\vDash^{l}_{L} \phi$.

3. Kripke-style semantics for Ls

We consider here algebraic Kripke-style semantics for L ($\in$ Ls).

Definition 3.1 ((Yang (2012), Algebraic Kripke frame) An *algebraic Kripke frame* is a structure **X** = (X, $\top$, $\bot$, t, f, $\leq$, $*$, $\rightarrow$) such that (X, $\top$, $\bot$, t, f, $\leq$, $*$, $\rightarrow$) is a linearly ordered residuated pointed bounded commutative monoid. The elements of **X** are called *nodes*.

Definition 3.2 (i) (Yang (2012), UL frame) A *UL frame* is an algebraic Kripke frame, where $*$ is conjunctive (i.e., $\bot * \top = \bot$) and left-continuous (i.e., whenever $\sup\{x_i : i \in I\}$ exists, $x * \sup\{x_i : i \in I\} = \sup\{x * x_i : i \in I\}$), and so its residuum $\rightarrow$ is defined as $x \rightarrow y := \sup\{z: x * z \leq y\}$ for all x, y $\in$ X.

(ii) The other frames are defined as follows: An *IUL*-frame is a UL-frame satisfying (dne). A *UML*-frame is a UL-frame satisfying (id). An *IUML*-frame is an IUL-frame satisfying (id) and (fp).

We call frames satisfying Definition 3.2 *L frames*. Definition 3.2 ensures that an L frame has a supremum w.r.t. $*$, i.e., for every x, y $\in$ X, the set $\{z: x * z \leq y\}$ has the supremum. **X** is said to be *complete* if $\leq$ is a complete order, i.e., where the join and meet $\vee$, $\wedge$ are supremum and infimum, respectively.

A *forcing* on an algebraic Kripke frame is a relation $\Vdash$

between nodes and propositional variables, and arbitrary formulas subject to the conditions below: for every propositional variable p,

(AHC) if $x \Vdash p$ and $y \leq x$, then $y \Vdash p$;
(min) $\bot \Vdash p$; and

for arbitrary formulas,

(t) $x \Vdash \mathbf{t}$ iff $x \leq t$;
(f) $x \Vdash \mathbf{f}$ iff $x \leq f$;
($\bot$) $x \Vdash \mathbf{F}$ iff $x = \bot$;
($\wedge$) $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$;
($\vee$) $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$;
(&) $x \Vdash \phi \,\&\, \psi$ iff there are $y, z \in X$ such that $y \Vdash \phi$, $z \Vdash \psi$, and $x \leq y * z$;
($\rightarrow$) $x \Vdash \phi \rightarrow \psi$ iff for all $y \in X$, if $y \Vdash \phi$, then $x * y \Vdash \psi$.

A forcing on an L frame is a forcing on an algebraic Kripke frame such that (max) for every atomic sentence p, $\{x : x \Vdash p\}$ has a maximum.

Definition 3.3 (i) (Algebraic Kripke model) An *algebraic Kripke model* is a pair $(\mathbf{X}, \Vdash)$, where $\mathbf{X}$ is an algebraic Kripke frame and $\Vdash$ is a forcing on $\mathbf{X}$.

(ii) (L model) A *L model* is a pair $(\mathbf{X}, \Vdash)$, where $\mathbf{X}$ is an L frame and $\Vdash$ is a forcing on $\mathbf{X}$. A L model $(\mathbf{X}, \Vdash)$ is said to be *complete* if **X** is a complete frame and $\Vdash$ is a forcing on **X**.

Definition 3.4 (Cf. Montagna & Sacchetti (2004)) Given an algebraic Kripke model (**X**, $\Vdash$), a node x of **X** and a formula ϕ, we say that *x forces* ϕ to express $x \Vdash \phi$. We say that ϕ is *true* in (**X**, $\Vdash$) if $t \Vdash \phi$, and that ϕ is *valid* in the frame **X** (expressed by $\mathbf{X} \vDash \phi$) if ϕ is true in (**X**, $\Vdash$) for every forcing $\Vdash$ on **X**.

For soundness and completeness for L, let $\vdash_L \phi$ be the theoremhood of ϕ in L. First we note the following lemma.

Lemma 3.5 (i) (Hereditary Lemma, HL) Let **X** be an algebraic Kripke frame. For any sentence ϕ and for all nodes $x, y \in \mathbf{X}$, if $x \Vdash \phi$ and $y \leq x$, then $y \Vdash \phi$.
(ii) Let $\Vdash$ be a forcing on an L frame, and ϕ a sentence. Then the set $\{x \in X : x \Vdash \phi\}$ has a maximum.

Proof: (i) Easy. (ii) See Lemma 2.11 in Montagna & Sacchetti (2003). □

By a *chain*, we mean a linearly ordered algebra. The next proposition connects algebraic Kripke semantics and algebraic semantics for L (cf. see Montagna & Sacchetti (2003; 2004)).

Proposition 3.6 (i) The $\{\top, \bot, t, f, \leq, *, \to\}$ reduct of a L-chain $\mathbf{A}$ is an L frame, which is complete iff $\mathbf{A}$ is complete.

(ii) Let **X** = (X, ⊤, ⊥, t, f, ≤, ∗, →) be an L frame. Then the structure **A** = (X, ⊤, ⊥, t, f, max, min, ∗, →) is a L-algebra (where *max* and *min* are meant w.r.t. ≤).

(iii) Let **X** be the {⊤, ⊥, t, f, ≤, ∗, →} reduct of a L-chain A, and let v be an evaluation in A. Let for every atomic formula p and for every x ∈ A, x ⊩ p iff x ≤ v(p). Then (**X**, ⊩) is an L model, and for every formula ϕ and for every x ∈ A, we obtain that: x ⊩ ϕ iff x ≤ v(ϕ).

(iv) Let (**X**, ⊩) be an L model, and let A be the L-algebra defined as in (ii). Define for every atomic formula p, v(p) = max{x ∈ X : x ⊩ p}. Then, for every formula ϕ, v(ϕ) = max{x ∈ X : x ⊩ ϕ}.

Proof: The proof is similar to that of Proposition 3.8 in Yang (2012). □

Proposition 3.7 (Soundness) If $\vdash_L$ ϕ, then ϕ is valid in every L frame.

Proof: We prove the validity of (DNE), (ID), and (FP) as examples:

(DNE) It suffices to assume x ⊩ (ϕ → **f**) → **f** and show x ⊩ ϕ. Suppose toward contradiction that x ⊮ ϕ. Since the sentence (ϕ → **f**) → **f** ↔ ϕ is a theorem in **IUL**, we have x ⊮ (ϕ → **f**) → **f**, a contradiction. Note that (ϕ → **f**) → **f** ↔ ϕ is a theorem in **IUL**.

(ID) We need to show that t ⊩ ϕ ↔ ϕ & ϕ. We prove the left-to-right direction. For this, it suffices to assume x $\Vdash$ ϕ and show x $\Vdash$ ϕ & ϕ. Assume x $\Vdash$ ϕ. Then, since x = x * x, using (&), we can obtain x $\Vdash$ ϕ & ϕ. The proof for its right-to-left is analogous.

(FP) We need to show that t $\Vdash$ **t** $\leftrightarrow$ **f**. We prove the left-to-right direction. For this, it suffices to assume x $\Vdash$ **t** and show x $\Vdash$ **f**. Assume x $\Vdash$ **t**. Then, since the sentence **t** $\leftrightarrow$ **f** is a theorem in **IUML**, we can obtain x $\Vdash$ **f**. The proof for its right-to-left is analogous.

The proof for the other cases is left to the interested reader. □

We can also obtain Proposition 3.7 as a corollary of the following proposition.

Proposition 3.8 Let **X** = (X, $\top$, $\bot$, t, f, $\leq$, $*$, $\rightarrow$) be an L frame, and let (L) be a name introduced in Definition 2.2 and $(L)_F$ be the corresponding property of an L frame introduced in Definition 3.2 (ii). Then, **X** $\vDash$ (L) iff **X** satisfies $(L)_F$.

Proof: It suffices to show that a linearly ordered UL-algebra is an L-algebra iff it satisfies the corresponding frame properties. As an example, we prove that **X** $\vDash$ (DNE) iff **X** satisfies $(DNE)_F$. By Proposition 3.6, it suffices to prove that a UL-algebra **A** is an IUL-algebra iff it satisfies $(DNE)_F$, i.e., (dne), This is immediate because frame properties are the same as algebraic properties. □

Theorem 3.9 (Strong completeness)

(i) L is strongly complete w.r.t. the class of all L-frames.

(ii) L is strongly complete w.r.t. the class of complete L-frames.

Proof: (i) and (ii) follow from Proposition 3.8 and Theorem 2.7, and from Proposition 3.8 and Corollary 2.9, respectively. □

4. Concluding remark

We investigated algebraic Kripke-style semantics for the weakening-free extensions of **UL** in Metcalfe & Montagna (2007). We proved soundness and completeness theorems. Note that Gabbay and Metcalfe also introduced other weakening-free extensions of **UL** in Gabbay & Metcalfe (2007). We can analogously introduce algebraic Kripke-style semantics for such systems. The investigation of these semantics is left to the interested reader.

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약화없는 퍼지 논리를 위한 대수적 크립키형 의미론

양 은 석

이 글에서 우리는 퍼지 논리들을 위한 크립키형 의미론을 다룬다. 보다 정확히 유니놈에 기반한 퍼지 논리 **UL**의 몇몇 약화없는 확장을 위한 대수적 크립키형 의미론을 소개한다. 이를 위하여 먼저 **UL**의 약화없는 확장 체계들을 소개하고 그에 상응하는 대수들을 정의한 후 이 체계들이 대수적으로 완전하다는 것을 보인다. 다음으로 이러한 체계들을 위한 크립키형 의미론을 소개하고 이를 대수적 의미론과 연관 짓는다.

주요어: 크립키형 의미론, 대수적 의미론, 다치 논리, 퍼지 논리