R, fuzzy R, and Algebraic Kripke-style Semantics* **

Eunsuk Yang

【Abstract】This paper deals with Kripke-style semantics for FR, a fuzzy version of R of Relevance. For this, first, we introduce FR, define the corresponding algebraic structures FR-algebras, and give algebraic completeness results for it. We next introduce an algebraic Kripke-style semantics for FR, and connect it with algebraic semantics. We furthermore show that such semantics does not work for R.

【Key Words】Kripke-style semantics, Algebraic semantics, Many-valued logic, Fuzzy logic, R, FR.

* 접수일자:2012.04.01 심사 및 수정 완료일:2012.05.20 게재확정일:2012.05.29

** This paper was supported by research funds of Chonbuk National University in 2012. I must thank the anonymous referees for their helpful comments.
1. Introduction

It is well known that many relevance logicians have had difficulties in providing binary relational Kripke-style semantics, i.e., semantics with binary accessibility relations, for relevance logics (see e.g. [3, 4]). To the best of my knowledge, any satisfactory such semantics for $\mathbf{R}$ has not yet been introduced. In this paper we show that such semantics can be provided for a fuzzy version of the system $\mathbf{R}$ of Relevance, although not $\mathbf{R}$ itself.

Actually, this is a free continuation of the paper [11]. In it the author provided algebraic Kripke-style semantics for Uninorm logic $\mathbf{UL}$. Here we introduce algebraic Kripke-style semantics for $\mathbf{FR}$, a fuzzy version of $\mathbf{R}$.\(^1\) For this, first, in Section 2 we introduce $\mathbf{FR}$, define the corresponding algebraic structures $\mathbf{FR}$-algebras, and give algebraic completeness results for it. In Section 3 we introduce an algebraic Kripke-style semantics for $\mathbf{FR}$, and connect them with algebraic semantics. We furthermore show that this semantics does not work for $\mathbf{R}$ (see Example 3.9).

For convenience, we shall adopt the notation and terminology similar to those in [5, 7, 8, 10], and assume familiarity with them (together with the results found in them).

2. The logic $\mathbf{FR}$ and its algebraic semantics

We base $\mathbf{FR}$ on a countable propositional language with

\(^1\) To see why algebraic Kripke-style semantics are interesting, see [12].
formulas \( \text{FOR} \) built inductively as usual from a set of propositional variables \( \text{VAR} \), binary connectives \( \rightarrow, \& \), \( \land \), \( \lor \), and constants \( f, t \), with defined connectives:\(^2\)

\[
\begin{align*}
\text{df1.} & \quad \neg \phi := \phi \rightarrow f \\
\text{df2.} & \quad \phi \leftrightarrow \psi := (\phi \rightarrow \psi) \land (\psi \rightarrow \phi).
\end{align*}
\]

We moreover define \( \phi_t := \phi \land t \). For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of \( \text{FR} \).

**Definition 2.1** \( \text{FR} \) consists of the following axiom schemes and rules:\(^3\)

A1. \( \phi \rightarrow \phi \) (self-implication, SI)
A2. \( (\phi \land \psi) \rightarrow \phi, (\phi \land \psi) \rightarrow \psi \) (\( \land \)-elimination, \( \land \)-E)
A3. \( ((\phi \rightarrow \psi) \land (\phi \rightarrow \chi)) \rightarrow (\phi \rightarrow (\psi \land \chi)) \) (\( \land \)-introduction, \( \land \)-I)
A4. \( \phi \rightarrow (\phi \lor \psi), \psi \rightarrow (\phi \lor \psi) \) (\( \lor \)-introduction, \( \lor \)-I)
A5. \( ((\phi \rightarrow \chi) \land (\psi \rightarrow \chi)) \rightarrow ((\phi \lor \psi) \rightarrow \chi) \) (\( \lor \)-elimination, \( \lor \)-E)
A6. \( (\phi \land (\psi \lor \chi)) \rightarrow ((\phi \land \psi) \lor (\phi \land \chi)) \) (\( \land \lor \)-distributivity, \( \land \lor \)-D)
A7. \( (\phi \& \psi) \rightarrow (\psi \& \phi) \) (\&-commutativity, \&-C)
A8. \( (\phi \& t) \leftrightarrow \phi \) (push and pop, PP)

\(^2\) Note that while \( \land \) is the extensional conjunction connective, \& is the intensional conjunction one.

\(^3\) A6, indeed, is redundant in \( \text{FR} \). But we introduce this in order to show that \( \text{R} \) is the \( \text{FR} \) omitting A13. Note that the system omitting both A6 and A13 is not \( \text{R} \) (cf see [1, 2, 4]).
A9. \((\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))\) (suffixing, SF)
A10. \((\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \& \psi) \to \chi)\) (residuation, RE)
A11. \(\phi \to (\phi \& \phi)\) (contraction, CR)
A12. \(\sim \sim \phi \to \phi\) (double negation elimination, DNE)
A13. \((\phi \to \psi)_t \lor (\psi \to \phi)_t\) (t-linearity, PL)
\(\phi \to \psi, \phi \vdash \psi\) (modus ponens, mp)
\(\phi, \psi \vdash \phi \land \psi\) (adjunction, adj).

A13 is the axiom scheme for linearity, and logics being complete w.r.t. linearly ordered (corresponding) algebras are said to be fuzzy logics (see e.g. [3]).

Note that the system \(\mathbf{R}\) is the \(\mathbf{FR}\) omitting A13. Note also that in \(\mathbf{R}\) (and so \(\mathbf{FR}\)), \(\phi \to \psi\) can be defined as \(\sim (\phi \& \sim \psi)\) (df3), and \(\phi \& \psi\) as \(\sim (\phi \to \sim \psi)\) (df4).

\textbf{Proposition 2.2} \(\mathbf{FR}\) proves:

(1) \((\phi \& (\psi \& \chi)) \leftrightarrow ((\phi \& \psi) \& \chi)\) (&-associativity, AS)
(2) \((\phi \& \psi) \to (\phi \& \psi)\)
(3) \((\phi \& (\psi \& \chi)) \leftrightarrow ((\phi \& \psi) \& (\phi \& \chi))\)
(4) \((\phi \to (\psi \lor \chi)) \leftrightarrow ((\phi \to \psi) \lor (\phi \to \chi))\)
(5) 
\(((\phi \to (\psi \lor \chi)) \land (\psi \to \chi)) \to (\phi \to \chi)\).

\textbf{Proof:} The proof for (1) to (3) is easy, just noting that in order to prove (3) we need A13 (cf. see [1]). We prove (4) and (5).

For the proof of (4), first note that in \(\mathbf{R}\), we can easily prove \((\phi \to (\psi \lor \chi)) \leftrightarrow (\phi \to \sim (\sim \psi \land \sim \chi))\) and \((\phi \to \sim (\sim \psi \land \sim \chi)) \leftrightarrow (\phi \to \sim (\sim \psi \land \sim \chi))\).
\[ \land \sim \chi \]) \leftrightarrow ((\phi \land (\sim \psi \land \sim \chi)) \rightarrow f). \] Then, using (3), we can prove ((\phi \land (\sim \psi \land \sim \chi)) \rightarrow f) \leftrightarrow \sim((\phi \land \sim \psi) \land (\phi \\
\land \sim \chi)), and using de Morgan laws, we get \sim((\phi \land \sim \psi) \land (\phi \land \sim \chi)) \leftrightarrow \sim(\phi \land \sim \psi) \lor \sim(\phi \land \sim \chi). Hence, by df3, we obtain (\phi \rightarrow (\psi \lor \chi)) \leftrightarrow ((\phi \rightarrow \psi) \lor (\phi \rightarrow \chi)), as required.

For the proof of (5), first note that in R, we can easily prove ((\phi \rightarrow \psi) \land (\psi \rightarrow \chi)) \rightarrow (\phi \rightarrow \chi) using (2). Then, since ((\phi \\
\rightarrow \chi) \land (\psi \rightarrow \chi)) \rightarrow (\phi \rightarrow \chi), we can obtain (((\phi \rightarrow \psi) \land (\psi \rightarrow \chi)) \lor (((\phi \rightarrow \psi) \land (\psi \rightarrow \chi))) \rightarrow (\phi \rightarrow \chi). Thus, using A6, we get ((\psi \rightarrow \chi) \land ((\phi \rightarrow \psi) \lor (\phi \rightarrow \chi))) \rightarrow (\phi \\
\rightarrow \chi). Hence, using (4), we can obtain that ((\phi \rightarrow (\psi \lor \chi)) \land (\psi \rightarrow \chi)) \rightarrow (\phi \rightarrow \chi), as wished. \square

Note that R does not prove (5) in Proposition 2.2 (see [5]).

In FR, f can be defined as \sim t. A theory over FR is a set T of formulas. A proof in a theory T over FR is a sequence of formulas whose each member is either an axiom of FR or a member of T or follows from some preceding members of the sequence using the two rules in Definition 2.1. T \models \phi, more exactly T \models_{FR} \phi, means that \phi is provable in T w.r.t. FR, i.e., there is a FR-proof of \phi in T. The relevant deduction theorem (RDT_t) for FR is as follows:

**Proposition 2.3** ([7]) Let T be a theory, and \phi, \psi formulas.  
(RDT_t) T \cup \{\phi\} \models \psi iff T \models_t \phi_t \rightarrow \psi.
For convenience, "~", "\&", "\lor", and "\rightarrow" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

The algebraic counterpart of \textbf{FR} is the class of \textit{FR-algebras}. Let \( x_t := x \& t \). They are defined as follows.

\textbf{Definition 2.4} (i) A \textit{pointed commutative residuated distributive lattice} is a structure \( A = (A, t, f, \&, \lor, *, \rightarrow) \) such that:

(I) \((A, \&, \lor)\) is a distributive lattice.

(II) \((A, *, t)\) is a commutative monoid.

(III) \( y \leq x \rightarrow z \) iff \( x * y \leq z \), for all \( x, y, z \in A \) (residuation).

(ii) (Dunn-algebras, [1, 2]) A \textit{Dunn-algebra} is a pointed commutative residuated distributive lattice satisfying:

(IV) \( x \leq x * x \) (contraction).

(V) \( (x \rightarrow f) \rightarrow f \leq x \) (double negation elimination).

(iii) (FR-algebras) A \textit{FR-algebra} is a Dunn-algebra satisfying:

(VI) \( t \leq (x \rightarrow y)_t \lor (y \rightarrow x)_t \) (pl).

Note that the class of Dunn-algebras characterizes the system \textbf{R}. Note also that Dunn-algebras are also called De Morgan monoids.

Additional (unary) negation and (binary) equivalence operations are defined as in Section 2.1: \( \sim x := x \rightarrow f \) and \( x \leftrightarrow y := (x \rightarrow y) \& (y \rightarrow x) \).

The class of all \textit{FR-algebras} is a variety which will be denoted by \textbf{FR}.
FR-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \land y = x$ or $x \land y = y$) for each pair $x, y$.

**Definition 2.5** (Evaluation) Let $A$ be an algebra. An $A$-evaluation is a function $v : \text{FOR} \to A$ satisfying: $v(\phi \to \psi) = v(\phi) \to v(\psi)$, $v(\phi \land \psi) = v(\phi) \land v(\psi)$, $v(\phi \lor \psi) = v(\phi) \lor v(\psi)$, $v(\phi \& \psi) = v(\phi) \ast v(\psi)$, $v(f) = f$, (and hence $v(\neg \phi) = \neg v(\phi)$ and $v(t) = t$).

**Definition 2.6** Let $A$ be a FR-algebra, $T$ a theory, $\phi$ a formula, and $K$ a class of FR-algebras.

(i) (Tautology) $\phi$ is a t-tautology in $A$, briefly an $A$-tautology (or $A$-valid), if $v(\phi) \geq t$ for each $A$-evaluation $v$.

(ii) (Model) An $A$-evaluation $v$ is an $A$-model of $T$ if $v(\phi) \geq t$ for each $\phi \in T$. By $\text{Mod}(T, A)$, we denote the class of $A$-models of $T$.

(iii) (Semantic consequence) $\phi$ is a semantic consequence of $T$ w.r.t. $K$, denoting by $T \models_K \phi$, if $\text{Mod}(T, A) = \text{Mod}(T \cup \{\phi\}, A)$ for each $A \in K$.

**Definition 2.7** (FR-algebra) Let $A$, $T$, and $\phi$ be as in Definition 2.6. $A$ is a *FR-algebra* iff whenever $\phi$ is FR-provable in $T$ (i.e. $T \vdash_{\text{FR}} \phi$), it is a semantic consequence of $T$ w.r.t. the set $\{A\}$ (i.e. $T \models_{\{A\}} \phi$), $A$ a FR-algebra. By $\text{MOD}^{(l)}(\text{FR})$, we denote the class of (linearly ordered) FR-algebras. Finally, we write $T \models^{(l)}_{\text{FR}} \phi$ in place of $T \models_{\text{MOD}^{(l)}(\text{FR})} \phi$. 
Note that since each condition for the FR-algebra has a form of equation or can be defined in equation (exercise), it can be ensured that the class of all FR-algebras is a variety.

We first show that classes of provably equivalent formulas form a FR-algebra. Let \( T \) be a fixed theory over \( \text{FR} \). For each formula \( \phi \), let \([\phi]_T\) be the set of all formulas \( \psi \) such that \( T \vdash_{\text{FR}} \phi \leftrightarrow \psi \) (formulas \( T \)-provably equivalent to \( \phi \)). \( A_T \) is the set of all the classes \([\phi]_T\). We define that \([\phi]_T \rightarrow [\psi]_T = [\phi \rightarrow \psi]_T\), \([\phi]_T \ast [\psi]_T = [\phi \& \psi]_T\), \([\phi]_T \land [\psi]_T = [\phi \land \psi]_T\), \([\phi]_T \lor [\psi]_T = [\phi \lor \psi]_T\), \( t = [t]_T \), and \( \bot_f = [f]_T \). By \( A_T \), we denote this algebra.

**Proposition 2.8** For \( T \) a theory over \( \text{FR} \), \( A_T \) is a \( \text{FR} \)-algebra.

**Proof:** Note that A1 to A6 ensure that \( \land \) and \( \lor \) satisfy (I) in Definition 2.4; that A7, A8, and AS ensure that \( \& \) satisfies (II); that A10 ensures that (III) holds; and that A11, A12, and A13 ensure that (IV), (V), and (VI), respectively, hold. It is obvious that \([\phi]_T \leq [\psi]_T \) iff \( T \vdash_{\text{FR}} \phi \leftrightarrow (\phi \land \psi) \) iff \( T \vdash_{\text{FR}} \phi \rightarrow \psi \). Finally recall that \( A_T \) is a \( \text{FR} \)-algebra iff \( T \vdash_{\text{FR}} \psi \) implies \( T \vdash_{\text{FR}} \phi \rightarrow \psi \), and observe that for \( \phi \) in \( T \), since \( T \vdash_{\text{FR}} t \rightarrow \phi \), it follows that \([t]_T \leq [\phi]_T \). Thus it is a \( \text{FR} \)-algebra. □

We next note that the nomenclature of the prelinearity condition is explained by the subdirect representation theorem below.

**Proposition 2.9** (Cf. [10]) Each \( \text{FR} \)-algebra is a subdirect
product of linearly ordered FR-algebras.

**Theorem 2.10** (Strong completeness) Let $T$ be a theory, and $\phi$ a formula. $T \vdash_{FR} \phi$ iff $T \models_{FR} \phi$ iff $T \models^l_{FR} \phi$.

**Proof:** (i) $T \vdash_{FR} \phi$ iff $T \models_{FR} \phi$. The left-to-right direction follows from definition. The right-to-left direction is as follows: from Proposition 2.8, we obtain $A_T \in \text{MOD}(FR)$, and for $A_T$-evaluation $v$ defined as $v(\psi) = [\psi]_T$, it holds that $v \in \text{Mod}(T, A_T)$. Thus, since from $T \models_{FR} \phi$ we obtain that $[\phi]_T = v(\phi) \geq t$, $T \vdash_{FR} t \rightarrow \phi$. Then, since $T \vdash_{FR} t$, by (mp) $T \vdash_{FR} \phi$, as required.

(ii) $T \models_{FR} \phi$ iff $T \models^l_{FR} \phi$. It follows from Proposition 2.9. □

3. Kripke-style semantics for FR

Here we consider algebraic Kripke-style semantics for FR.

**Definition 3.1** (Algebraic Kripke frame) An algebraic Kripke frame is a structure $X = (X, t, f, \leq, *, \rightarrow)$ such that $(X, t, f, \leq, *, \rightarrow)$ is a linearly ordered residuated pointed commutative monoid. The elements of $X$ are called nodes.

**Definition 3.2** (FR frame) A FR frame is an algebraic Kripke frame, where $x = (x \rightarrow f) \rightarrow f$, and $*$ is contractive, i.e., $x \leq x * x$. 


An evaluation or forcing on an algebraic Kripke frame is a relation $\models$ between nodes and propositional variables, and arbitrary formulas subject to the conditions below: for every propositional variable $p$,

(AHC) if $x \models p$ and $y \leq x$, then $y \models p$; and

for arbitrary formulas,

(t) $x \models t$ iff $x \leq t$;
(f) $x \models f$ iff $x \leq f$;
($\land$) $x \models \phi \land \psi$ iff $x \models \phi$ and $x \models \psi$;
($\lor$) $x \models \phi \lor \psi$ iff $x \models \phi$ or $x \models \psi$;
($\&$) $x \models \phi \& \psi$ iff there are $y, z \in X$ such that $y \models \phi$, $z \models \psi$, and $x \leq y \ast z$;
($\rightarrow$) $x \models \phi \rightarrow \psi$ iff for all $y \in X$, if $y \models \phi$, then $x \ast y \models \psi$.

**Definition 3.3** (i) (Algebraic Kripke model) An algebraic Kripke model is a pair $(X, \models)$, where $X$ is an algebraic Kripke frame and $\models$ is a forcing on $X$.

(ii) (FR model) A FR model is a pair $(X, \models)$, where $X$ is a FR frame and $\models$ is a forcing on $X$.

**Definition 3.4** (Cf. [9]) Given an algebraic Kripke model $(X, \models)$, a node $x$ of $X$ and a formula $\phi$, we say that $x$ forces $\phi$ to express $x \models \phi$. We say that $\phi$ is true in $(X, \models)$ if $t \models \phi$, and
that \( \phi \) is \textit{valid} in the frame \( X \) (expressed by \( X \) models \( \phi \)) if \( \phi \) is true in \( (X, \models) \) for every forcing \( \models \) on \( X \).

For soundness and completeness for \( \text{FR} \), let \( \vdash_{\text{FR}} \phi \) be the theoremhood of \( \phi \) in \( \text{FR} \). First we note the following lemma.

\textbf{Lemma 3.5} (Hereditary Lemma, HL) Let \( X \) be an algebraic Kripke frame. For any sentence \( \phi \) and for all nodes \( x, y \in X \), if \( x \models \phi \) and \( y \leq x \), then \( y \models \phi \).

\textbf{Proof:} Easy. \( \square \)

\textbf{Proposition 3.6} (Soundness) If \( \vdash_{\text{FR}} \phi \), then \( \phi \) is valid in every FR frame.

\textbf{Proof:} We prove the validity of A11 as an example: it suffices to show that if \( x \models \phi \), then \( x \models \phi \land \phi \). Assume \( x \models \phi \). Then, since \( x \leq x \ast x \), using \( (\land) \), we can obtain \( x \models \phi \land \phi \), as required.

The proof for the other cases is left to the interested reader. \( \square \)

By a \textit{chain}, we mean a linearly ordered algebra. The next proposition connects algebraic Kripke semantics and algebraic semantics for \( \text{FR} \) (cf. see [9]).

\textbf{Proposition 3.7} (i) The \( \{t, f, \leq, \ast, \rightarrow\} \) reduct of a \( \text{FR} \)-chain \( A \) is a FR frame.
(ii) Let $X = (X, t, f, \leq, *, \rightarrow)$ be a FR frame. Then the structure $A = (X, t, f, \text{max}, \text{min}, *, \rightarrow)$ is a FR-algebra (where $\text{max}$ and $\text{min}$ are meant w.r.t. $\leq$).

(iii) Let $X$ be the $\{t, f, \leq, *, \rightarrow\}$ reduct of a FR-chain $A$, and let $v$ be an evaluation in $A$. Let for every atomic formula $p$ and for every $x \in A$, $x \models p$ iff $x \leq v(p)$. Then $(X, \models)$ is a FR model, and for every formula $\phi$ and for every $x \in A$, we obtain that: $x \models \phi$ iff $x \leq v(\phi)$.

**Proof:** The proof for (i) and (ii) is easy. For the proof of (iii), see Proposition 3.8 in [10]. □

**Theorem 3.8** (Strong completeness) FR is strongly complete w.r.t. the class of all FR-frames.

**Proof:** It follows from Proposition 3.7 and Theorem 2.10. □

Let an $R$ frame $X$ be an FR frame on a partially ordered monoid in place of a linearly ordered monoid, let an evaluation or forcing $\models$ on an $R$ frame be the same as that on a FR frame, and let $(X, \models)$ be an $R$ model. Then, at first glance, $(X, \models)$ seems to be a model for $R$. But actually it is not. The following example verifies it.

**Example 3.9** An $R$ model $(X, \models)$ validates Proposition 2.2 (5), i.e., $t \models ((\phi \rightarrow (\psi \lor \chi)) \land (\psi \rightarrow \chi)) \rightarrow (\phi \rightarrow \chi)$.  

Proof: By ($\rightarrow$) and ($\land$), we assume $x \vdash (\phi \rightarrow (\psi \lor \chi))$ and $x \vdash \psi \rightarrow \chi$, and show $x \vdash \phi \rightarrow \chi$. For this last, we further assume $y \vdash \phi$ and show $x \ast y \vdash \chi$. By the suppositions and ($\rightarrow$), we have $x \ast y \vdash \psi \lor \chi$, therefore $x \ast y \vdash \psi$ or $x \ast y \vdash \chi$ by ($\lor$). Let $x \ast y \vdash \psi$. Then, since $x \vdash \psi \rightarrow \chi$, by ($\rightarrow$) we obtain $x \ast (x \ast y) \vdash \chi$, therefore $(x \ast x) \ast y \vdash \chi$ by the associativity of $\ast$. Then, since $x \leq x \ast x$, using Lemma 3.5, we get $x \ast y \vdash \chi$. □

This sentence is not a theorem of $R$ but a theorem of $FR$. Thus this model is not for $R$.

4. Concluding remark

We investigated algebraic Kripke-style semantics for $FR$, a fuzzy version of $R$. We proved soundness and completeness theorems. But we did not provide algebraic Kripke-style semantics for $R$. This is an open problem left in this paper.
References


전북대학교 철학과
Department of Philosophy, Chonbuk National University
eunsyang@jbnu.ac.kr
R, fuzzy R, and Algebraic Kripke-style Semantics

양 은 석


주요어: R, FR, (대수적) 크립키형 의미론, 대수적 의미론, 다치 논리, 퍼지 논리