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Standard Completeness for the Weak Uninorm Mingle Logic **WUML***

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【Abstract】Fixed-point conjunctive left-continuous idempotent uninorms have been introduced (see e.g. [2, 3]). This paper studies a system for such uninorms. More exactly, one system obtainable from **IUML** (Involutive uninorm mingle logic) by dropping involution (INV), called here **WUML** (Weak Uninorm Mingle Logic), is first introduced. This is the system of fixed-point conjunctive left-continuous idempotent uninorms and their residua with weak negation. Algebraic structures corresponding to the system, i.e., WUML-algebras, are then defined, and algebraic completeness is provided for the system. Standard completeness is further established for **WUML** and **IUML** in an analogy to that of **WNM** (Weak nilpotent minimum logic) and **NM** (Nilpotent minimum logic) in [4].

【Key Words】(substructural) fuzzy logic, fuzzy logic, idempotent uninorm (based) logic

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Ⅰ. Introduction

In general a function n : [0, 1] → [0, 1] is called *negation* function (briefly a negation) if and only if (iff) n is non-increasing and satisfies n(0) = 1 and n(1) = 0. A negation n satisfying n(n(x)) ≥ x for all x ∈ [0, 1] is said to be a *weak negation*; and a weak negation n is called a *strong negation* (or *involutive negation*) if n further satisfies n(n(x)) = x for all x ∈ [0, 1]. In strong negations on [0, 1], n(x) can be defined as 1 - x, i.e., n(x) = 1 - x, called the standard negation.

For the past 11 or 12 years idempotent uninorms, in particular, fixed-point conjunctive left-continuous idempotent uninorms, have been introduced (see e.g. [2, 3, 7]). Furthermore, the logic of the conjunctive left-continuous idempotent uninorm with strong negation and identity e = n(e) = ½, i.e., **IUML** (Involutive uninorm mingle logic), has been recently introduced in [8]. But as far as the author knows, any other systems, which are logics of conjunctive left-continuous idempotent uinorms with weak negations (in place of strong negation) and identity e ∈ (0, 1] introduced in [2, 3] have not yet been studied. We shall here introduce a system for such uninorms. More exactly, we first introduce **WUML** (Weak uninorm mingle logic) as a generalization of **IUML** having weak negation instead of strong negation. We then define the corresponding algebraic structures, WUML-algebras, and prove algebraic completeness for it. We further establish standard completeness for **WUML** and **IUML** in an analogy to that of **WNM** (Weak nilpotent minimum logic) and

NM (Nilpotent minimum logic) in [4].

For convenience, we shall adopt the notation and terminology similar to those in [1, 4, 6, 8], and assume being familiar with them (together with results found in them).

Ⅱ. Syntax

We base the weak uninorm mingle logic **WUML** on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$, and

df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define $\phi_{\mathbf{t}}$ as $\phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of **WUML** as a (substructural) fuzzy logic.

Definition 2.1 **WUML** consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\phi\to\psi)\wedge(\phi\to\chi)) \to (\phi\to(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi\to\chi)\wedge(\psi\to\chi)) \to ((\phi\vee\psi)\to\chi)$ ($\vee$-elimination, $\vee$-E)

A6. $\phi \to \mathbf{T}$ (verum ex quolibet, VE)

A7. $\mathbf{F} \to \phi$ (ex falso quadlibet, EF)

A8. $(\phi \& \psi) \to (\psi \& \phi)$ (&-commutativity, &-C)

A9. $(\phi \& \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)

A10. $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ (suffixing, SF)

A11. $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \& \psi) \to \chi)$ (residuation, RE)

A12. $(\phi \to \psi)_{\mathbf{t}} \vee (\psi \to \phi)_{\mathbf{t}}$ (prelinearity, PL)

A13. $(\phi \& \phi) \leftrightarrow \phi$

A14. $\mathbf{t} \leftrightarrow \mathbf{f}$

A15. $(\phi \to \psi) \vee ((\phi \to \psi) \to (\sim\phi \wedge \psi))$

A16. $(\phi \to \psi) \to (\sim\phi \vee \psi)$

A17. $(\phi \to \sim\psi) \to ((\phi \& \psi) \to (\phi \wedge \psi))$

A18. $(\phi \& \psi) \to ((\phi \vee \psi) \to (\phi \& \psi))$

A19. $\sim\mathbf{T} \to \mathbf{F}$

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj).

The system having A1 to A12, mp, and adj is **UL** (Uninorm logic); the **UL** having A13 is **UML** (Uninorm mingle logic); and the **UML** with A14 and (INV) $\sim\sim\phi \to \phi$ is **IUML**. These were introduced in [8]. We from now on call a system satisfying A1 to A14 a *fixed-point uninorm mingle logic*, briefly, a FUML.

Proposition 2.2 WUML proves:

(1) $(\phi \mathbin{\&} (\psi \mathbin{\&} \chi)) \leftrightarrow ((\phi \mathbin{\&} \psi) \mathbin{\&} \chi)$ (&-associativity, AS)

(2) $\phi \vee {\sim}\phi$ (excluded middle, EM)

(3) ${\sim}(\phi \wedge {\sim}\phi)$ (non-contradiction, NC)

(4) $(\phi \to \psi) \to ((\phi \to \psi) \leftrightarrow ({\sim}\phi \vee \psi))$

(5) $({\sim}\phi \wedge \psi) \to (\phi \to \psi)$

(6) $((\phi \to \psi) \to ({\sim}\phi \wedge \psi)) \to (({\sim}\phi \wedge \psi) \to (\phi \to \psi))$

(7) $((\phi \to \psi) \to ({\sim}\phi \wedge \psi)) \to ((\phi \to \psi) \leftrightarrow ({\sim}\phi \wedge \psi))$

(8) $(\phi \to {\sim}\psi) \vee (\phi \mathbin{\&} \psi)$

(9) $(\phi \to {\sim}\psi) \to ((\phi \mathbin{\&} \psi) \leftrightarrow (\phi \wedge \psi))$

(10) $(\phi \mathbin{\&} \psi) \to ((\phi \mathbin{\&} \psi) \leftrightarrow (\phi \vee \psi))$

(11) ${\sim}\mathbf{T} \leftrightarrow \mathbf{F}$.

In **WUML**, **f** can be defined as ~**t** and vice versa. A *theory* over **WUML** is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of **WUML** or a member of T or follows from some preceding members of the sequence using the rules mp and adj. $T \vdash \phi$, more exactly $T \vdash_{\mathbf{WUML}} \phi$, means that ϕ is *provable* in T w.r.t. **WUML**, i.e., there is a **WUML**-proof of ϕ in T. The **t**-deduction theorem (DT_t) for **WUML** is as follows:

Proposition 2.3 Let T be a theory, and ϕ, ψ formulas. $T \cup \{\phi\} \vdash_{\mathbf{WUML}} \psi$ iff $T \vdash_{\mathbf{WUML}} \phi_t \to \psi$.

Proof: See [9]. □

A theory T is *inconsistent* if $T \vdash \mathbf{F}$; otherwise it is *consistent*.

For convenience, "~", "$\wedge$", "$\vee$", and "$\rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

III. Semantics

Suitable algebraic structures for **WUML** are obtained as a subvariety of the variety of commutative residuated lattices in the sense of e.g. [5].

Definition 3.1 A *pointed bounded commutative residuated lattice* is a structure **A** = $(A, \top, \bot, \top_t, \bot_f, \wedge, \vee, *, \rightarrow)$ such that:

(I) $(A, \top, \bot, \wedge, \vee)$ is a bounded lattice with top element $\top$ and bottom element $\bot$.

(II) $(A, *, \top_t)$ satisfies for some $\top_t$ and for all $x, y, z \in A$,

(a) $x * y = y * x$ (commutativity)

(b) $\top_t * x = x$ (identity)

(c) $x * (y * z) = (x * y) * z$ (associativity).

(III) $y \leq x \rightarrow z$ iff $x * y \leq z$, for all $x, y, z \in A$ (residuation).

$(A, *, \top_t)$ satisfying (II-b, c) is a *monoid*. Thus $(A, *, \top_t)$ satisfying (II-a, b, c) is a commutative monoid. To define the above lattice we may take in place of (III) a family of equations as in [6]. Using $\rightarrow$ and $\bot_f$ we can define $\top_t$ as $\bot_f \rightarrow \bot_f$, and ~ as in (df1). Then, WUML-algebra whose class characterizes

WUML is defined as follows.

Definition 3.2 (WUML-algebra) A *WUML-algebra* is a pointed bounded commutative residuated lattice satisfying the conditions: for all x, y,

(pl) $\top_t \leq (x \rightarrow y)_{\top t} \vee (y \rightarrow x)_{\top t}$,

(id) $x * x = x$,

(fp) $\top_t = \bot_f$,

(w1) $\top_t \leq (x \rightarrow y) \vee ((x \rightarrow y) \rightarrow ({\sim}x \wedge y))$,

(w2) $x \rightarrow y \leq {\sim}x \vee y$,

(w3) $x \rightarrow {\sim}y \leq (x * y) \rightarrow (x \wedge y)$,

(w4) $x * y \leq (x \vee y) \rightarrow (x * y)$, and

(w5) ${\sim}\top = \bot$.

Note that UL-algebras are pointed bounded commutative residuated lattices satisfying (pl); UML-algebras are UL-algebras satisfying (id); and IUML-algebras are UML-algebras satisfying (fp) and ${\sim\sim}x \leq x$.

WUML-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function $v : FOR \rightarrow \mathcal{A}$ satisfying:

$v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$,

$v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$,

$v(\phi \vee \psi) = v(\phi) \vee v(\psi)$,
$v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$,
$v(\mathbf{F}) = \bot$,
$v(\mathbf{f}) = \bot_{\mathbf{f}}$,

(and hence $v(\sim\phi) = \sim v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = \top_{\mathbf{t}}$).

Definition 3.4 Let $\mathcal{A}$ be a WUML-algebra, T a theory, ϕ a formula, and K a class of WUML-algebras.

(i) (Tautology) ϕ is a $\top_t$-*tautology* in $\mathcal{A}$, briefly an $\mathcal{A}$-*tautology* (or $\mathcal{A}$-*valid*), if $v(\phi) \geq \top_{\mathbf{t}}$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-*model* of T if $v(\phi) \geq \top_{\mathbf{t}}$ for each $\phi \in$ T. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_{\mathsf{K}} \phi$, if $\mathrm{Mod}(T, \mathcal{A}) = \mathrm{Mod}(T \cup \{\phi\}, \mathcal{A})$ for each $\mathcal{A} \in \mathsf{K}$.

Definition 3.5 (**WUML**-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is a ***WUML**-algebra* iff whenever ϕ is WUML-provable in T (i.e. $T \vdash_{\mathbf{WUML}} \phi$), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. $T \vDash_{\{A\}} \phi$), $\mathcal{A}$ a WUML-algebra. By *$MOD^{(l)}$(**WUML**)*, we denote the class of (linearly ordered) **WUML**-algebras. Finally, we write $T \vDash^{(l)}_{\mathbf{WUML}} \phi$ in place of $T \vDash_{MOD^{(l)}(\mathbf{WUML})} \phi$.

Note that since each condition for the WUML-algebra has a form of equation or can be defined in equation (exercise), it can be ensured that the class of all WUML-algebras is a variety.

Let **A** be a WUML-algebra. We first show that classes of provably equivalent formulas form a WUML-algebra. Let T be a fixed theory over **WUML**. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that T $\vdash_{\mathbf{WUML}}$ $\phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \rightarrow [\psi]_T = [\phi \rightarrow \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \,\&\, \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $\bot = [\mathbf{F}]_T$, $\top = [\mathbf{T}]_T$, $\top_{\mathbf{t}} = [\mathbf{t}]_T$, and $\bot_{\mathbf{f}} = [\mathbf{f}]_T$. By $\boldsymbol{A_T}$, we denote this algebra.

Proposition 3.6 For T a theory over **WUML**, $\mathbf{A}_T$ is a **WUML**-algebra.

Proof: Note that A1 to A7 ensure that $\wedge$ and $\vee$ satisfy (I) in Definition 3.1; that AS, A8, A9 ensure that & satisfies (II); that A11 ensures that (III) holds; and that A12 to A19 ensure that the conditions in Definition 3.2 hold. It is obvious that $[\phi]_T \leq [\psi]_T$ iff T $\vdash_{\mathbf{WUML}}$ $\phi \leftrightarrow (\phi \wedge \psi)$ iff T $\vdash_{\mathbf{WUML}}$ $\phi \rightarrow \psi$. Finally recall that $\mathbf{A}_T$ is a **WUML**-algebra iff T $\vdash_{\mathbf{WUML}}$ ψ implies T $\vDash_{\mathbf{WUML}}$ ψ, and observe that for ϕ in T, since T $\vdash_{\mathbf{WUML}}$ $\mathbf{t} \rightarrow \phi$, it follows that $[\mathbf{t}]_T \leq [\phi]_T$. Thus it is a **WUML**-algebra. □

We next note that the nomenclature of the prelinearity condition is explained by the subdirect representation theorem below.

Proposition 3.7 (Cf. [10]) Each WUML-algebra is a subdirect product of linearly ordered WUML-algebras.

Theorem 3.8 (Strong completeness) Let T be a theory, and ϕ a formula. $T \vdash_{\mathbf{WUML}} \phi$ iff $T \vDash_{\mathbf{WUML}} \phi$ iff $T \vDash^{1}_{\mathbf{WUML}} \phi$.

Proof: (i) $T \vdash_{\mathbf{WUML}} \phi$ iff $T \vDash_{\mathbf{WUML}} \phi$. Left to right follows from definition. Right to left is as follows: from Proposition 3.6, we obtain $\mathbf{A}_T \in$ MOD(L), and for $\mathbf{A}_T$-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in$ Mod(T, $\mathbf{A}_T$). Thus, since from $T \vDash_{\mathbf{WUML}} \phi$ we obtain that $[\phi]_T = v(\phi) \geq \top_{\mathbf{t}}$, $T \vdash_{\mathbf{WUML}} \mathbf{t} \rightarrow \phi$. Then, since $T \vdash_{\mathbf{WUML}} \mathbf{t}$, by (mp) $T \vdash_{\mathbf{WUML}} \phi$, as required.

(ii) $T \vDash_{\mathbf{WUML}} \phi$ iff $T \vDash^{1}_{\mathbf{WUML}} \phi$. It follows from Proposition 3.7. □

Ⅳ. Uninorms and their residua

In this section, using *1, 0, and some* 1_t, *and* 0_f in the real unit interval [0, 1], we shall express $\top$, $\bot$, $\top_{\mathbf{t}}$, and $\bot_{\mathbf{f}}$, respectively. We also define standard WUML-algebras and uninorms on [0, 1].

Definition 4.1 A WUML-algebra is *standard* iff its lattice reduct is [0, 1].

In standard WUML-algebras the monoid operator * is a uninorm.

Definition 4.2 A *uninorm* is a function $\circ : [0, 1]^2 \to [0, 1]$ such that for some $1_t \in [0, 1]$ and for all x, y, z $\in$ [0, 1]:

(a) $x \circ y = y \circ x$ (commutativity),
(b) $x \circ (y \circ z) = (x \circ y) \circ z$ (associativity),
(c) $x \leq y$ implies $x \circ z \leq y \circ z$ (monotonicity), and
(d) $1_t \circ x = x$ (identity).

The function $\circ$ satisfying (1-identity) $1_t = 1$ is a *t-norm*. A uninorm $\circ$ is said to be *square increasing* if it satisfies (square-increasingness) $x \leq x \circ x$, for all x $\in$ [0, 1]; *square decreasing* if it satisfies (square-decreasingness) $x \circ x \leq x$, for all x $\in$ [0, 1]; and *idempotent* if it is both square increasing and square decreasing. A uninorm is called *conjunctive* if $0 \circ 1 = 0$; *disjunctive* if $0 \circ 1 = 1$; and *fixed-point* if $1_t = 0_f$.

The left-continuity property of conjunctive uninorms is important in the sense that it gives a residuated implication and so plays an important role in standard completeness proof of **WUML** as in t-norm based logics such as **MTL**. $\circ$ is said to be *residuated* iff there is $\to : [0, 1]^2 \to [0, 1]$ satisfying (residuation) on [0, 1]. Given a uninorm $\circ$, *residuated implication* $\to$ determined by $\circ$ is defined as $x \to y := \sup\{z: x \circ z \leq y\}$ for all x, y $\in$ [0, 1]. Then, we can show that for any uninorm $\circ$, $\circ$ and its residuated implication $\to$ form a residuated pair iff $\circ$ is conjunctive and left-continuous in both arguments (see Proposition 5.4.2 [9]).

It is clear that the operator * of any standard UL-algebra is a conjunctive uninorm with identity $\top_t$ and residuum $\rightarrow$; conversely any residuated uninorm gives rise to a UL-algebra as follows: if $\circ$ is a uninorm with residuum $\rightarrow$ and identity 1_t, then for any $0_f \in [0, 1]$, $([0, 1], 1, 0, 1_t, 0_f, \min, \max, \circ, \rightarrow)$ is a standard UL-algebra (see Proposition 17 in [8]).

We finally state some known interesting facts related to conjunctive left-continuous idempotent uninorms and their residua.

Fact 4.3 (i) ([2]) A binary operator $\circ$ is a conjunctive left-continuous idempotent uninorm with identity element $1_t \in (0, 1]$ iff there is a left-continuous uniquely determined non-increasing function $n : [0, 1] \rightarrow [0, 1]$ with $n(1_t) = 1_t$ and $n(n(x)) \geq x$ for $x \in [0, 1]$, such that for all $x, y \in [0, 1]$:

$$x \circ y = \begin{cases} \min(x, y) & \text{if } y \leq n(x), \\ \max(x, y) & \text{otherwise.} \end{cases}$$

(ii) ([8]) Let $\mathbf{A}_s = ([0, 1], 1, 0, ½, ½, \min, \max, \circ_s, \rightarrow_s)$, where:

$$x \circ_s y = \begin{cases} \min(x, y) & \text{if } x + y \leq 1, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

ϕ is valid in all standard IUML-algebras iff ϕ is valid in the IUML-algebra $\mathbf{A}_s$.

The operator $\circ_s$ in (ii) of Fact 4.3 is an example of (i) satisfying strong negation. We introduce an unknown further example of (i) with weak negation.

Example 4.4 Given a *fixed-point weak* negation n, i.e., a negation n satisfying: for all $x \in [0, 1]$,

(a) $n(1_t) = 1_t$,

(b) $n(n(x)) \geq x$, and

(c) $n(0) = 1$ and $n(1) = 0$,

we can construct a conjunctive left-continuous idempotent uninorm $\bigcirc$ given by, for all $x, y \in [0, 1]$:

$$x \bigcirc y = \begin{cases} \min(x, y) & \text{if } y \leq n(x), \\ \max(x, y) & \text{otherwise.} \end{cases}$$

We call uninorms in Fact 4.3 (ii) and Example 4.4 IUML- and WUML-uninorms, respectively.

Left-continuity of the above uninorms ensures that the corresponding residuated implications can be obtained. The following are the known facts.

Fact 4.5 (i) ([3]) Consider a conjunctive left-continuous idempotent uninorm $\bigcirc$ with a negation n. Then its residuated implication $\rightarrow$ is given by

$$x \rightarrow y = \begin{cases} \max(n(x), y) & \text{if } x \leq y, \\ \min(n(x), y) & \text{otherwise.} \end{cases}$$

(ii) ([3, 7]) Consider an involutive negation n_s. Then the residiated implication of the corresponding conjunctive left-continuous idempotent uninorm $\bigcirc_s$ is given by

$$x \rightarrow_s y = \begin{cases} \max(1\text{-}x, y) & \text{if } x \leq y, \\ \min(1\text{-}x, y) & \text{otherwise.} \end{cases}$$

The residiated implication of the corresponding conjunctive left-continuous idempotent uninorms with the negation in Example 4.4 is obtained as in Fact 4.5.

IUML is the system satisfying (ii) in Fact 4.3 and so (ii) in Fact 4.5. Any system satisfying (i) in Fact 4.3 corresponds to a FUML, i.e., a fixed-point uninorm mingle logic. But, as far as the author knows, such systems having weak negations in place of strong negation have not yet been introduced.

V. Standard completeness

We here provide *standard* completeness results for **WUML** and **IUML** in an analogy to that of **WNM** and **NM** in [4]. First note that weak and strong negations are defined as in Section 1. Moreover,

Definition 5.1 ([4]) Given a bounded linearly ordered set $(C, \leq, \top, \bot)$, a non-increasing function $n : C \to C$ is said to be *symmetric* w.r.t. the identity mapping if it satisfies:

1. if $x \in n(C)$, then $n(x) = y$ implies $x = n(y)$, and
2. if $x \notin n(C)$, then

(i) n is constant in the interval $[x, n^2(x)]$ with value $n(x)$, and

(ii) for any $y > n(x)$, it is $n(y) < x$, i.e., $n(x)$ is a discontinuity point on the right with $n(n(x)^-) = n^2(x)$ and $n(n(x)^+) < x$.

We call linearly ordered WUML- and IUML-algebras *WUML- and IUML-chains*. We first note that given a bounded chain C, $n : C \to C$ is a weak negation iff it is non-increasing and symmetric w.r.t. the identity mapping (see Proposition A.3 in [4]). As a consequence, we then give **WUML**- and **IUML**-chains with weak and strong negations, respectively, as follows.

Proposition 5.2 (i) Given a weak negation n on [0, 1], we can define a residuated pair of operations $\bigcirc_n$, $\rightarrow_n$ such that ([0, 1], $\bigcirc_n$, $\rightarrow_n$, 1, 0, 1_t, 0_f) is a **WUML**-chain with negation n.

(ii) For any strong negation n_s on [0, 1], we can define a residuated pair of operations $\bigcirc_{ns}$, $\rightarrow_{ns}$ such that ([0, 1], $\bigcirc_{ns}$, $\rightarrow_{ns}$, 1, 0, 1_t, 0_f) is an **IUML**-chain with negation n_s.

Proof: (i) Let n be the n in Example 4.4. We first note that given a weak negation n on [0, 1], a uninorm $\bigcirc_n$ for **WUML**, called the *weak Gödel uninorm*, is defined as in (i) of Example 4.4. Then, given a weak negation n on [0, 1], we can easily prove that:

(a) $\bigcirc_n$ has residuum defined by

$x \rightarrow_n y =$ max(n(x), y) if $x \leq y$;
min(n(x), y) otherwise (Weak Gödel u-implication), where u-implication is an abbreviation of *uninorm-implication*,

(b) ([0, 1], $\bigcirc_n$, $\rightarrow_n$, 1, 0, 1_t, 0_f) is a WUML-algebra, and

(c) the corresponding negation is n.

Since the axioms of **WUML** are valid in a WUML-algebra, this algebra is a **WUML**-algebra as well.

Note that the condition $x \leq n(y)$ is *symmetric* in the sense that it is equivalent to $y \leq n(x)$ because $x \leq n(y)$ implies $n^2(y)$

$\leq$ n(x), and y $\leq$ $n^2(y)$.

(ii) Proof of the case of strong negation is analogous to that of the negation in (i), i.e., given a strong negation n_s on [0, 1], we can easily prove that:

(a) $\circ_{ns}$ has residuum defined by

$x \rightarrow_{ns} y$ = max(1-x, y) if x $\leq$ y;

min(1-x, y) otherwise (Strong Gödel u-implication), where u-implication is an abbreviation of uninorm-implication,

(b) ([0, 1], $\circ_{ns}$, $\rightarrow_{ns}$, 1, 0, ½, ½) is an IUML-algebra, and

(c) the corresponding negation n is the strong one. □

Fact 5.3 ([4]) (1) Weak negation functions on the real unit interval [0, 1] are not all isomorphic, and yet (2) strong negation functions on it are.

Then because of Fact 5.3 we can say that

Proposition 5.4 (i) Weak negation functions on [0, 1] are not all isomorphic so that **WUML**-chains defined by WUML-uninorms are not.

(ii) Strong negation functions on [0, 1] are all isomorphic so that **IUML**-chains defined by IUML-uninorms are.

Theorem 5.5 (Weak standard completeness) (i) $\vdash_{\mathbf{WUML}}$ ϕ iff ϕ is a tautology in all standard **WUML**-algebras.

(ii) $\vdash_{\mathbf{IUML}}$ ϕ iff ϕ is a tautology w.r.t. the standard

IUML-algebra ([0, 1], $\bigcirc_{ns}$, $\rightarrow_{ns}$, 1, 0, ½, ½).

Proof: Each proof of (i) and (ii) is analogous to that of Theorems 3 and 4, respectively, in [4].

(i) Soundness is obvious. For completeness, let $\nvdash_{\mathbf{WUML}} \phi$. We show that there is a **WUML**-chain C and an evaluation v on C such that $v(\phi) < 1_t$. We prove the case n_C. Let n_C be the negation in C and take X as the finite subset of C consisting of all the values $v(\psi)$, $n_C(v(\psi))$, and $n_C(n_C(v(\psi)))$ for all subformulas ψ of ϕ, plus 0_C, 1_C, and 1_{tC} (= 0_{fC}). Let

$$X \cap n_C(C) = \{0_C = a_0 < a_1 < \cdots < 1_{tC} = 0_{fC} < \cdots < a_m = 1_C\},$$

where $1_{tC} = a_{m/2}$ if m is even and otherwise $1_{tC} = a_{(m+1)/2}$. Let $f : X \rightarrow [0, 1]$ be an ordered mapping ($x < y$ implies $f(x) < f(y)$) such that $f(a_i) = i/m$. Define on [0, 1] the weak negation function n as follows: taking $1_t = f(1_{tC})$, and letting $1_t^- = 1_t - 1/m$,

$$n(x) = \begin{cases} 1 - x & \text{if } x \in \{i/m: 0 \leq i \leq m\} \text{ and } x \neq 1_t, \\ 1_t & \text{if } 1_t^- < x \leq 1_t, \\ (m-i-1)/m & \text{otherwise, i.e.,} \\ & \text{if } x \in (i/m, (i+1)/m) \text{ where } i/m \neq 1_t^-, \end{cases}$$

It is clear that n is a fixed-point weak negation on [0, 1] and f is a morphism w.r.t. minimum, maximum, implication, and negation. Note that we take $1_t = f(1_{tC})$. Thus, by defining the evaluation $v'(p) = v(f(p))$ for any propositional variable p occurring in ϕ, we get $v'(\phi) = v(f(\phi)) < 1_t$, as desired.

(ii) Soundness is obvious. For completeness, let $\nvdash_{\mathbf{IUML}} \phi$. Then there is an **IUML**-chain C and an evaluation v on C such that $v(\phi) < 1_t$. Let n_{sC} be the strong negation in C and take X as the finite subset of C, whose number of elements being odd, consisting of all the values $v(\psi)$ and $n_{sC}(v(\psi))$ for all subformulas ψ of ϕ together with 0_C, 1_C, and 1_{tC} ($= 0_{fC}$). Suppose that X has m + 1 elements and let $X = \{0_C = a_0 < a_1 < \cdots < 1_{tC} = 0_{fC} < \cdots < a_m = 1_C\}$, where $1_{tC} = a_{m/2}$ and m is even. Now let $f(a_i) = i/m$, $0 \leq i \leq m$. It is clear that f, as a mapping from X to the set $\{i/m: 0 \leq i \leq m\}$, is a morphism w.r.t. minimum, maximum, implication, and negation, and we take $1_t = f(1_{tC})$. Thus, defining $v'(p) = v(f(p))$ for any propositional variable p occurring in ϕ, we get $v'(\phi) = v(f(\phi)) < 1_t$, as required. Thus, $\vdash_{\mathbf{IUML}} \phi$ iff ϕ is a tautology in all standard **IUML**-algebras.

Finally note that Proposition 5.4 (ii) ensures that ϕ is a tautology in all standard **IUML**-algebras iff ϕ is a tautology in the standard $([0, 1], \circ_{ns}, \rightarrow_{ns}, 1, 0, ½, ½)$. □

Furthermore, we can show strong standard completeness for **WUML** and **IUML**.

Theorem 5.6 (Strong standard completeness) For **L** $\in$ {**WUML**, **IUML**}, let T be a theory over **L**, and ϕ a formula. $T \vdash_{\mathbf{L}} \phi$ iff ϕ is true in each standard **L**-model of T.

Proof: Its proof is analogous to that of Theorem 4.2.17 in [6]. □

Ⅵ. Concluding remark

In this paper we introduced the system **WUML**, which is a generalization of **IUML** having weak negation in place of strong negation. After defining the corresponding algebraic structures, we provided algebraic completeness for it. Furthermore, we established standard completeness for **WUML** and **IUML**.

In fact we can also introduce systems for other fixed-point conjunctive left-continuous idempotent uninorms. We shall investigate this in some subsequent paper.

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WUML의 표준적 완전성

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이 논문은 uninorm의 한 체계인 WUML(Weak Uninorm Mingle Logic)을 다룬다. 먼저 WUML을 도입하고, 그 체계에 상응하는 대수적 구조를 정의한다. 그런 다음 그 체계의 대수적 완전성을 증명한다. 끝으로 WUML과 IUML의 표준적 완전성을 증명한다.

주요어: (준구조) 퍼지 논리, 퍼지 논리, 멱등 유니놈 논리