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Algebras and Semantics for Dual Negations* †

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Dunn investigated algebras and semantics for negations in non-classical logics. This paper extends his investigation to dual negations, more exactly to duals to the negations in Dunn [3, 5]. I first survey and systematize the algebras of dual negations, i.e., (self-dual) subminimal negation, dual Galois negations, dual minimal negation, wB (or dual intuitionistic) negation, (self-dual) De Morgan negation, and (self-dual) ortho negation, based on partially ordered sets. I next provide dual-perp semantics for these (dual) negations. I finally give representations for them by using dual-perp semantics.

【Key Words】(algebras of) dual negations, dual-perp semantics, representations.

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1. Introduction

Dunn [1 - 6] investigated algebras and semantics for relationships among main properties of negation as they emerged in various non-classical logics (as well as classical logic). He [3, 5] especially discussed the subminimal negation, the Galois negations, the minimal negation, the De Morgan negation, the intuitionistic negation, and the ortho negation on partially ordered sets (posets); moreover, the preminimal negation and the Ockham negation on distributive lattices (see [5]).

Yang [8] has recently investigated weak Boolean (wB) algebra, and called its complementation "wB complementation" and corresponding negation "wB negation". One interesting point to note is that while wB negation satisfies the condition for subminimal negation, it does not belong to the varieties of negation above. (Note that subminimal negation may be regarded as basic one in the sense that every negation Dunn investigated there requires its condition, i.e., contraposition.) We note that wB negation is dual to the intuitionistic negation in the sense that the former accepts duals to the conditions for the latter, i.e., while the latter accepts constructive double negation and absurdity, the former classical double negation and triviality, respectively.

Then a natural concern arises about each dual of the negations Dunn investigated (and the relationship between negations and dual negations). We shall give several representations for these dual negations by providing dual-perp semantics.

In connection with dual-perp semantics for dual negations, it will be interesting to note that the dual-perp $\top$ for the wB (dual intuitionistic) negation requires "reflexivity", which is dual to "irreflexivity" the perp $\perp$ for the intuitionistic negation requiring. We recall that in [5] Dunn stated that "as soon as one allows the presence of inconsistent information states, as must be done to accommodate relevance and other "paraconsistent logics", we must give up irreflexivity". (Note that Dunn [3, 5] did not include such negations behaving paraconsistently in his treatments of negation, and that as dual to the intuitionistic negation the wB negation behaves paraconsistently.) We are willing to give it up (for dual negations). To emphasize this, we shall think of $\top$ as "(negation-)paracompatibility".

So, we discuss here relationships among main properties of dual negation as they may emerge in various (possible) non-classical logics. More exactly, we first survey and systematize the algebras of dual negations, which negations we shall call the (self-dual) subminimal negation, the dual Galois negations, the dual minimal negation, wB (or the dual intuitionistic) negation, the (self-dual) De Morgan negation, and the (self-dual) ortho negation, based on posets. We next provide dual-perp semantics for these (dual) negations by considering the dual-perp $\top$ as (negation-)paracompatibility rather than incompatibility.

While we may obtain ortho negation from intuitionistic negation by adding to it classical double negation, we may get dual ortho negation from wB negation by adding to it constructive double negation. However, in (the frameworks of) these semantics, where we only have negation and binary consequence (see below), we can distinguish neither dual ortho negation from ortho negation nor (self-dual) ortho negation from Boolean negation. In addition, while in general lattices we can distinguish (self-dual) ortho negation from Boolean negation (cf. see [7]), we can not distinguish dual ortho negation from ortho negation because they are still the same in such lattices.

For convenience, we shall frequently call each corresponding complementation just its logical name negation; disjunction and conjunction with respect to join and meet, respectively. We get the idea of our discussion from several works of Dunn's, especially from [3, 5].

2. Varieties of dual negation

We survey and systematize varieties of dual negation for (possible) non-classical logics without stating varieties of negation investigated by Dunn [3, 5]. We just assume familiarity with them.

Let P = (P, ≤) be a poset with bounds 0 and 1, i.e., $\forall x(0 \leq x)$ and $\forall x(x \leq 1)$. For convenience, we think of the elements of P to be propositions, and read a ≤ b as "the proposition a implies the proposition b". Thus, when we have this interpretation in mind we shall call P a consequence poset as in [5]. We may regard 0 as a proposition that implies every proposition, and 1 as a proposition that is implied by every proposition. We sometimes assume that the poset is outfitted with the lattice operations meet ∧ and join ∨, (and thought of as conjunction and disjunction), satisfying the following laws, where a ∧ b is the greatest lower bound of a and b, and a ∨ b is the least upper bound of a and b:

(1) $(a \wedge b) \leq a$, $(a \wedge b) \leq b$
(2) if $a \leq b$ and $a \leq c$, then $a \leq (b \wedge c)$
(3) $a \leq (a \vee b)$, $b \leq (a \vee b)$
(4) if $a \leq c$ and $b \leq c$, then $(a \vee b) \leq c$.

We also sometimes assume the following distributive law:

(5) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

The *(self-dual) subminimal negation*, which has been investigated by Hazen (see [5]), is an operation on a consequence poset satisfying the following:

(6) if $a \leq b$, then $-b \leq -a$ (contraposition).

Let $(P, \leq)$ and $(Q, \leq')$ be posets. A pair of functions $(\neg, -)$ is called a *dual Galois connection* between P and Q if and only if (iff) $- : P \Rightarrow Q$, $\neg : Q \Rightarrow P$, and

(7) if $x_1 \leq x_2$, then $-x_2 \leq' -x_1$,
(8) if $y_1 \leq' y_2$, then $\neg y_2 \leq \neg y_1$,
(9) for $x \in P$, $\neg - x \leq x$,
(10) for $y \in Q$, $-\neg y \leq' y$.

Proposition 1 The definition of dual Galois connection above is equivalent to

(11) for $x \in P$, $y \in Q$, $\neg y \leq x$ iff $-x \leq' y$ (dual Galois property).

Proof: To derive (11) from the definition of a dual Galois connection, we assume that $-x \leq' y$. Then $\neg y \leq \neg - x$ by (8), and so $\neg y \leq x$ by (9) and transitivity. By using (7) and (10), the other direction can be proved.

Conversely, all of the parts of the d efinition of a dual Galois connection can be derived from (11). First, since $-x \leq' -x$, $\neg - x \leq x$ and thus we obtained (9). For (7), we assume $x_1 \leq x_2$. Then, $\neg - x_1 \leq x_2$ by (9) and transitivity. Thus, by (11) we obtained $-x_2 \leq' -x_1$. Similarly, by using the other half of (11), we can derive (8) and (10). □

Similarly to the Galois negations, when the above two posets are identical, we speak of *dual Galois (connected) negations* or *dual split negation.*

The *dual minimal negation* is the subminimal negation with

(12) $-\,-a \leq a$ (classical double negation).

Instead, we may require simply

(13) if $-a \leq b$, then $-b \leq a$ (classical contraposition).

Note that the dual minimal negation is a pair of dual Galois negations where $- = \urcorner$.

The *wB (or dual intuitionistic) negation* is obtained by adding to dual minimal negation a property that says that excluded middles are implied by everything, i.e., $a \vee -a = 1$. Since we do not necessarily have disjunction (lattice join), this can be defined officially as follows:

(14) if $b \leq a$ and $-b \leq a$, then $a = 1$ (triviality).

The *(self-dual) De Morgan negation* arises from the dual minimal negation by adding

(15) $a \leq -\,-a$. (constructive double negation).

Since dual minimal negation satisfies the converse, this amounts to requiring that $a = -\,-a$ (period two). The (self-dual) De Morgan negation may be defined by (13) and

(16) if $a \leq -b$, then $b \leq -a$(constructive contraposition)

in place of (15). The *(self-dual) ortho negation* is obtained by adding to the dual minimal negation both the properties of the triviality and the constructive double negation.

Let the underlying poset be a bounded distributive lattice D = (D, ∧, ∨) with the least element 0 and the greatest element 1. The *(self-dual) Ockham negation* — : D ⇒ D is a dual endomorphism:

(17) $-(x \vee y) = -x \wedge -y$,
(18) $-(x \wedge y) = -x \vee -y$.

Moreover, the (self-dual) ortho negation is just the well-known Boolean negation.

3. Paracompatibility C^{-}_{p}: Algebraic version

In [5] (the paper of generalized ortho negation), Dunn discussed "incompatibility" in algebraic and semantic versions, based on Gottblatt's intuition of a relation of incompatibility (called "orthogonality" or "perp") between states. Incompatibility requires the properties antitonicity, irreflexivity, and symmetry. As he mentioned in it, under the hypothesis of irreflexivity "there are no contradictions in the sense of propositions that imply both of two contradictory propositions", and thus "there is a deficiency in the algebraic version of negation as incompatibility".

One interesting point to state is that with respect to dual negations (a relation of) incompatibility requires dually reflexivity. We note that with respect to dual negation(s) there may be the case that any proposition is compatible with both the negation of its negation (in the sense that double negated proposition of a — —a can be included in a itself) and its negation, i.e., a may be compatible with both — —a and —a (see (25) below). So in place of the name "incompatibility" we shall call this relation *negation-paracompatibility*, briefly *paracompatibility*, in the sense that a may be compatible with each negation of both —a and a. (We borrow the name "paracompatibility" from the "paraconsistency" (used in paraconsistent logic) in the sense that there may be a consistent theory T such that both $\phi \in T$ and $-\phi \in T$.)

In an analogy to negations in [5] we define dual negations in virtue of paracompatibility C^{-}_{p}. We form a complete join semi-lattice as a structure $(A, \leq, \vee, C^{-}_{p})$, where $\leq$ is a partial order, every $B \subset A$ has a least upper bound (l.u.b) $b \in A$, i.e., $\forall x(\forall y \in B(y \leq x)$ implies $b \leq x)$, $\vee$ is an operation such that $\vee B$ is the l.u.b of a set, and C^{-}_{p} is a binary relation on A that is isotone in each of its positions with respect to $\leq$:

(19) if $a\ C^{-}p\ x$ and $a \leq b$, then $b\ C^{-}p\ x$,
(20) if $x\ C^{-}p\ a$ and $a \leq b$, then $x\ C^{-}p\ b$.

Now we characterize — (as [self-dual] subminimal negation), even in the absence of $\vee$, by:

(21) $-a \leq x$ iff $a\, C^{-}p\, x$.

Then, by substituting $-a$ for x, it follows from (21) that (22) $a\, C^{-}_{p} -a$. From (22) we can prove contraposition as follows: let $a \leq b$. By (21), $b\, C^{-}_{p} -a$ only if $-b \leq -a$. Thus, since $b\, C^{-}_{p} -a$ by $a \leq b$, (22), and (19), $-b \leq -a$.

For dual Galois negations, we define another negation operation as follows:

(23) $\neg a \leq x$ iff $x\, C^{-}p\, a$.

By (21) and (23), we can have dual Galois (connected) negations:

(24) $-a \leq b$ iff $\neg b \leq a$.

Thus using (21) and (23), we obtain that $-a \leq b$ iff $a\, C^{-}_{p}\, b$ iff $\neg b \leq a$.

For dual minimal negation, we add to a subminimal negation a condition that C^{-}_{p} is symmetric. By using symmetry, we can prove classical double negation with respect to $-$: $-a \leq -a$. Then, by (21), $a\ C^{-}_{p} -a$. By symmetry, $-a\, C^{-}_{p}\, a$, and thus $- -a \leq a$ by (21). (Note that by using contraposition and classical double negation, (13) can be obtained.)

For wB (or dual intuitionistic) negation, we add to a dual minimal negation a condition such that

(25) if a $C^{-}p$ a and —a $C^{-}p$ a, then a = 1 ($C^{-}p$-triviality).

We prove triviality as follows: let b ≤ a and —b ≤ a. Then, by the second and (21), b C^{-}_{p} a. Thus, a C^{-}_{p} a by (19) and the first. Now, since a C^{-}_{p} —a by (22), —a C^{-}_{p} a by symmetry. Hence, by (25), a = 1, as desired.

For (self-dual) De Morgan negation and (self-dual) ortho negation, we first define an inverse (relation) of C^{-}_{p} C^{--1}_{p} as follows:

(26) a $C^{--1}p$ b iff a ≤ —b.

Now we add to a dual minimal negation and a wB negation, respectively, a condition such that

(27) a $C^{--1}p$ —a ($C^{--1}p$ —a = a).

Then constructive double negation follows directly from (27) and (26).

4. Dual-perp semantics for dual negations

Depending on [3, 5], where Dunn provided perp semantics for negations in non-classical logics, we give semantics for dual negations. For convenience, we shall adopt the notations, definitions, and results related with perp semantics for negations found in [3, 5], and assume familiarity with them.

As in [3, 5], in examining the semantic treatments of dual negation we suppose that we are dealing with so called "UCLA propositions", i.e., sets of verifying states. Thus, with any given sentence ϕ we can associate with it as its interpretation $|\phi| = \{\alpha : \alpha \models \phi\}$. (This allows us to talk of negation directly as an operation on sets, and omits the "middle man" of syntax.)[1)]

Let an *information frame* **Fr** be a structure (U, $\sqsubseteq$), where U is a non empty set (of "states") and $\sqsubseteq$ is a partial order on U. Note that we should require of the sets (that are pro positions) the following backward hereditary condition:

(28) (BHC) if $\alpha \in A$ and $\beta \sqsubseteq \alpha$, then $\beta \in A$.

We define a *dual*-perp *frame* as a structure **Fr** = (U, $\sqsubseteq$, $\top$), where (U, $\sqsubseteq$) is an information frame and $\top$ is a binary relation on U that is antitonic in each of its positions with respect to $\sqsubseteq$:

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1) Note that the treatments of dual negation are similar to those of negation, and thus they can be illustrated by the following semantic clauses:

($\top^{-}$) $\chi \models -\phi$ iff $\exists\alpha(\alpha \top \chi$ and $\alpha \not\models \phi)$.

See [3] for the definitions of negation in terms of UCLA propositions and corresponding definitions of operation on UCLA propositions. Analogously, we can give definitions of dual negation in terms of UCLA propositions and corresponding definitions of operation on such propositions.

(29) if $\alpha \sqsubseteq \beta$ and $\beta \top \chi$, then $\alpha \top \chi$ ($\top$-left-antitone);
(30) if $\alpha \sqsubseteq \beta$ and $\chi \top \beta$, then $\chi \top \alpha$ ($\top$-right-antitone).

The *propositions P(F)* on an information frame are the backward hereditary subsets A of U which satisfy (28).

(31) $A^{\top} = \{\chi : A \top \chi\}$, where $A \top \chi := \exists \alpha(\alpha \top \chi$ and $\alpha \not\in A)$.

Dually to (31), let us write that $\chi \top A := \exists \alpha(\chi \top \alpha$ and $\alpha \not\in A)$. We denote this dual negation by ${}^{\top}A = \{\chi : \chi \top A\}$. Note that if $\top$ is symmetric, $A^{\top} = {}^{\top}A$, and if not, we get two negations $A^{\top}$ and ${}^{\top}A$. As Dunn states "$A \subseteq B^{\perp}$ iff $B \subseteq {}^{\perp}A$" as the Galois property in [5], we can dually say that they satisfy

(32) $A^{\top} \subseteq B$ iff ${}^{\top}B \subseteq A$.[2)]

This is the dual Galois property (11). Moreover, in an analogy to those in [5] we can state the semantic conditions for dual negations as follows: (note that the conditions below for each dual negation correspond to the requirements for each dual negation in algebraic version.)

Negation	Dual-perp Semantics
Subminimal	$-A = A^{\top}$

2) (32) based on the above definitions was first suggested by Dunn, see [6]. I must say that I considered (32) based on different definitions but these had some problems, and that the referee suggested the above definitions.

dual Galois	$\neg A = {}^{\top}A$
dual Minimal	$\top$ symmetric
wB(dual Intuitionistic)	$\top$ reflexive, sym.
De Morgan	$A = A^{\top\top}$, sym.
Ortho	$A = A^{\top\top}$, sym., ref.

Remark 1 Note that in an analogy to dual-perp semantics, we can provide C^{-}_{p} semantics as follows: first, define a *C^{-}_{p} frame* as a structure $(U, \sqsubseteq, C^{-}_{p})$, where $(U, \sqsubseteq)$ is an information frame and C^{-}_{p} is a binary relation on U that is isotone in each of its positions with respect to $\sqsubseteq$. $A^{C\text{-}p} = \{\chi : A\ C^{-}_{p}\ \chi\}$, where $A\ C^{-}_{p}\ \chi := \exists\alpha(\alpha\ C^{-}_{p}\ \chi$ and $\alpha \not\in A)$. ${}^{C\text{-}p}A = \{\chi : \chi\ C^{-}_{p}\ A\} = \{\chi : \exists\alpha(\chi\ C^{-}_{p}\ \alpha$ and $\alpha \not\in A)\}$. Next, add to a C^{-}_{p} frame the dual-perp semantic conditions for each dual negation, but with C^{-}_{p} in place of $\top$. For example, add to a C^{-}_{p} frame "$-A = A^{C\text{-}p}$" as the C^{-}_{p} semantic condition for (self-dual) subminimal negation.

Note that in section 5 we prove several representations just by using dual-perp semantics. However, it must be ensured that in an analogy to representations by dual-perp semantics we can provide representations for dual negations by C^{-}_{p} semantics.

5. Representations

In this section we give representations for dual negations by using dual-perp semantics. Let P = (P, $\leq$) be a bounded poset. The elements of P are to be thought of as propositions, and a $\leq$ b is to be read "the proposition a implies the proposition b", and so we shall call P a *(binary) consequence poset* when we have this interpretation in mind.

Let P = (P, $\leq$, —, $\neg$) be a bounded poset with two order-inverting mappings — and $\neg$. We call this a *(self-dual) subminimal two negation (consequence) poset.* Note that to get a single (self-dual) subminimal negation we may assume that the two operations — and $\neg$ are the same. By a *two dual-perp frame*, we mean a structure (U, $\sqsubseteq$, $\top_1$, $\top_2$), where each (U, $\sqsubseteq$, $\top_1$) and (U, $\sqsubseteq$, $\top_2$) is a dual-perp frame. Every two dual-perp frame gives rise to a (self-dual) subminimal two negation poset, where we let P be the set of backward hereditary subsets of U, we let $\leq$ be $\subseteq$ among those backward hereditary subsets, and for A $\in$ P, we define —A = $A^{\top 1}$, and $\neg$A = $A^{\top 2}$. 0 is the empty set and 1 is U. We call this the *full two dual-perp negation poset determined by the frame.*

Note that if $\top_1$ and $\top_2$ are converses and $\top_1$ is symmetric, then $\top_1$ = $\top_2$. So, we can collapse a two dual-perp frame into a single dual-perp frame, and similarly a two dual-perp negation poset determined by the frame to a single dual-perp negation poset.

A cone C is *proper* in case C is not all of P. We define the *canonical two dual-perp frame determined by **P*** to be a structure $F_{can}^{2\top} = (U_c, \sqsubseteq_c, \top^{-}, \top^{\neg})$, where U_c is the set of proper cones of P, $C_1 \sqsubseteq_c C_2$ iff $C_2 \subseteq C_1$, for cones C_1 and C_2, $C_1 \top^{-} C_2$ iff $\forall a(-a \notin C_2$ implies $a \in C_1)$, and $C_1 \top^{\neg} C_2$ iff $\forall a(\neg a \notin C_2$ implies $a \in C_1)$. The canonical isomorphism is as follows:

(33) $h(a) = \{C : C$ is a proper cone of P and $a \in C\}$.

We first state representation for (self-dual) subminimal two negation. (Note that based on this we prove the remaining representations.)

Proposition 2 (Representation for [self-dual] subminimal two negation) Let $\mathbf{P} = (P, \leq, -, \neg)$ be a (self-dual) subminimal two negation consequence poset. Then there is a two dual-perp frame so that P can be isomorphically embedded in the full (self-dual) subminimal two dual-perp negation poset determined by that frame.

Proof: We sketch the proof here, observing that we must show

(34) $h(-a) = h(a)^{\top}$, i.e.,
(35) $-a \in C$ iff $\exists C'(C' \top^{-} C$ and $a \notin C')$.

Left to right is immediate. By contraposition, we prove right to left. We assume that $-a \notin C$ and show that $\forall C'(C' \top^{-} C$ implies $a \in C')$. Note that $C_1 \top^{-} C_2$ iff $\forall x(-x \notin C_2$

implies $x \in C_1$). Thus, it follows from $-a \not\in C$ that $a \in C'$, as required. □

Proposition 3 (Representation for dual Galois negations) Let $\mathbf{P} = (P, \leq, -, \neg)$ be a bounded consequence poset with dual Galois negations $(-, \neg)$. Then there is a two dual-perp frame $(U, \sqsubseteq, \top_1, \top_2)$, where $\top_1$ and $\top_2$ are converses, so that $\mathbf{P}$ can be isomorphically embedded in the full two dual-perp negation poset determined by that frame.

Proof: We can easily prove that if $(-, \neg)$ are dual Galois negations, then $\top^{\neg}$ is the converse of $\top^{-}$. Thus we just show that $C_1 \top^{-} C_2$ iff $C_2 \top^{\neg} C_1$. (i) For left to right suppose that $C_1 \top^{-} C_2$. Then, we can say that (*) $\forall x(-x \not\in C_2$ implies $x \in C_1)$. To show $C_2 \top^{\neg} C_1$, we assume that $x \not\in C_2$ and show that $\neg x \in C_1$. Since $-\neg x \leq x$, $-\neg x \not\in C_2$ and so by (*) $\neg x \in C_1$, as wanted. (ii) Right to left is similar to (i). □

Let $\top_1$ and $\top_2$ be symmetric. Then, as we noted above, if they are converse, they are just the same relation. Thus, we need not distinguish $-$ from $\neg$. So, in the remaining representations we work with a structure $\mathbf{P} = (P, \leq, -)$.

Proposition 4 (Representation for dual minimal negation) Let $\mathbf{P} = (P, \leq, -)$ be a bounded consequence poset with dual minimal negation $-$. Then there is a dual-perp frame

(U, ⊑, ⊤), where ⊤ is symmetric, so that P can be isomorphically embedded in the full two dual-perp negation poset determined by that frame.

Proof: To show that there is a dual-perp frame with ⊤ symmetric, reconfigure (P, ≤, —) as a dual Galois negations poset (P, ≤, —, ⌐), where — = ⌐. Then $\top^{-} = \top^{\neg}$, and from the proof of Proposition 3 we know that they are converses of each other. □

Proposition 5 (Representation for wB (or dual intuitionistic) negation) Let P = (P, ≤, —) be a bounded consequence poset with wB (or dual intuitionistic) negation —. Then there is a dual-perp frame (U, ⊑, ⊤), where ⊤ is reflexive and symmetric, so that P can be isomorphically embedded in the full dual-perp negation poset determined by that frame.

Proof: To prove this, we use U_c = the set of *complete* cones, where a cone C is *complete* in case for any x, x ∈ C or —x ∈ C, instead of the set of all proper cones U.

First, to show that there is a dual-perp frame with ⊤ reflexive, it suffices to note that ∀x(x ∈ C or —x ∈ C) implies ∀x(—x ∉ C only if x ∈ C), i.e., C $\top^{-}$ C. For symmetry of ⊤, see Proposition 3.

Next, under the canonical frame and isomorphism above, we must show (35) above. In the constructions, following

Corollary 4 in [5], we show that those cones exist. That is, in the following two places, we show that the required cone is complete.

The first place is in showing that h is one-one. Let us assume that $a \neq b$ and thus without loss of generality that $a \nleq b$. Then, consider the non-empty set of all cones that contain a and do not contain b, and order it by inclusion. Since every chain has an upper bound, by Zorn's Lemma there is a maximal cone C that contains a and does not contain b: suppose toward contradiction that there is x such that $x \notin C$ and $-x \notin C$. Then, since $C \cup [x)$ and $C \cup [-x)$ extend C and contain a, we must have $b \in C \cup [x)$ and $b \in C \cup [-x)$. Since $b \notin C$, we have $x, -x \leq b$. Then, by triviality $b = 1$, which is contrary to the supposition that $a \nleq b$. Thus, C is complete.

The other place is in the argument for the right to left of (35). First, as in Proposition 2, We can assume that $-a \notin C$ and show that $\forall C'(C' \top^{-} C$ implies $a \in C')$. Next, we need to show that C′ is complete. Otherwise, there is y such that $y \notin C'$ and $-y \notin C'$. This implies that $a \nleq y$ and $a \nleq -y$. Then, by the same argument as $a \nleq b$, just with y, −y, and C′ in place of x, −x, and C, respectively, we can show that C′ is complete. □

Note that De Morgan negation is self-dual. By requiring that $a = --a$ (period two) and correspondingly that $A = A^{\top\top}$ ("closed" sets), as in De Morgan negation in [3], we can give representation for (self-dual) De Morgan negation, just with — in place of ⌐ in Corollary 5 in it, as follows.

Proposition 6 (Representation for [self-dual] De Morgan negation) Let **P** = (P, ≤, —) be a bounded consequence poset with (self-dual) De Morgan negation —. Then there is a dual-perp frame (U, ⊑, ⊤), where ⊤ is symmetric, so that **P** can be isomorphically embedded in the full (self-dual) De Morgan dual-perp negation poset determined by that frame.

Note that if a (≠ 1) ∉ C (= [1), i.e., the principal cone determined by 1), by completeness —a (= 1) ∈ C and so — —a (= 0) ∉ C. Then, since — —a = a by period two, a = 0. This implies that a ≤ b, —b only if a = 0, i.e., *absurdity* holds in it, and that a (≠ 1) ∈ C only if C is *improper*, i.e., C is all of P, since a = — —a = 0 ∈ C. Note also that for Cone Separation Principle (i.e., a ≰ b only if there is a cone C with a ∈ C and b ∉ C), given absurdity, [a) must be negation-consistent, i.e., there is no x ∈ [a) such that —x ∈ [a). For otherwise, for some x, a ≤ x and a ≤ —x, and so a is the least element 0. But then a ≤ b.

Proposition 7 (Representation for [self-dual] ortho negation) Let **P** = (P, ≤, —) be a bounded consequence poset with (self-dual) ortho negation —. Then there is a dual-perp frame (U, $\sqsubseteq$, $\top$), where $\top$ is reflexive and symmetric, so that P can be isomorphically embedded in the full (self-dual) De Morgan dual-perp negation poset determined, by that frame.

Proof: To prove this, we just note that we use U_c = the set of *negation-consistent and complete* cones. Then, in an analogy to Proposition 6, we can prove this. □

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부록 | Abstracts

ARTICLE ABSTRACTS

Algebras and Semantics for Dual Negations

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Dunn investigated algebras and semantics for negations in non-classical logics. This paper extends his investigation to dual negations, more exactly to duals to the negations in Dunn [3, 5]. I first survey and systematize the algebras of dual negations, i.e., (self-dual) subminimal negation, dual Galois negations, dual minimal negation, wB (or dual intuitionistic) negation, (self-dual) De Morgan negation, and (self-dual) ortho negation, based on partially ordered sets. I next provide dual-perp semantics for these (dual) negations. I finally give representations for them by using dual-perp semantics.

【Key Words】 (algebras of) dual negations, dual-perp semantics, representations.