Routley-Meyer Semantics for E-R, Ee-R, Eec-R

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Abstract

In this paper we investigate the relevance logic E-R of the Entailment E without the reductio (R), and its extensions Ee-R, Eec-R: Ee-R is the E-R with the expansion (e) and Eec-R the Ee-R with the chain (c). We give completeness for each E-R, Ee-R, and Eec-R by using Routley-Meyer semantics.

1. Introduction

Among relevance systems, E of Entailment has been conside important from early on in the study of relevance logics. Note tha the favorite of Anderson and Belnap: they are the beginners of r logics (see [1]). Most of all, this system is important in the sense implication, i.e., so called entailment, shows `necessity' and `r which are the necessary and sufficient conditions of logical imp note that the strict implication of modal logic just shows the 'ne and that the relevant implication of R just the 'relevance'. (cf. [1

Many (algebraic) papers say that this system has the same connectives as the classical logic with respect to conjunction, disjunction, and negation (see [5]). However, exactly speaking (from the algebraic point of view), this is not true. When we consider `De Morgan lattices' (or `quasi-Boolean algebras') that characterize **E** with respect to those connectives, we find that **E** has one unnatural axiom scheme, i.e. the reductio (R), in the sense that this is not the part characterized by those algebras.[1)]

Thus, in [14] we investigated some neighbors of the relevance **T** of Ticket Entailment without the reductio (R), i.e., the **T** w (T-R), and its extensions the T-R with the expansion (e) (Te-R), the Te-R with the chain (c) (Tec-R). We also gave completeness for these systems by using Routley-Meyer (RM) semantics. Note that in it we already explained why relevance systems without R, e.g., T-R, can be formally (or algebraically) interesting; that **E** is not decidable and that the decidability of **EW**, i.e., the **E** without the contraction (W), still remains as an open question (see [3, 4, 8, 13]). Note also that e is important (and useful) in decidability and c is important when we consider any system to be many-valued logic, especially infinite-valued logic, (consider the intuitionist propositional logic H and the Dummett's LC of its infinite-valued extension that can be regarded as the **H** with c).

We, however, postpone to investigate the decidability of the neighbors of **E** with e and c (as many-valued logics) to another occasion because we need too much space to complete it in this paper. Instead, we just show that the same idea as in [14] can be applied to the system **E**. Namely, we investigate the **E** without R (E-R), the E-R with e (Ee-R), and the Ee-R with c (Eec-R). Note that Routley and Meyer [10, 11] investigated RM semantics for Positive Entailment $\mathbf{E}^+$ and **E** and gave completeness for each system. However, they did not do that for

1) To understand the meaning of "characterize" (or " characteristic"), see this passage in [7]: "Given a matrix M, let Taut(M) (the "tautologies of M") be the set of sentences that take designated values for every interpretation in underlying algebra. Given a unary assertional logic L, a matrix M is said to be characteristic for L whenever for all sentences ϕ, $\vdash_L \phi$ iff $\phi \in$ Taut(M)."

the above E-systems, and we have not yet found any literature that investigated them. So, we give RM semantics for each E-R, Ee-R, and Eec-R, and thus its completeness. This means that the neighbors of **E** with e and c can be semantically complete.

For convenience, by *E(ec)-R* we shall ambiguously express E-R, Ee-R, Eec-R all together, if we do not need distinguish them, but context should determine which system is intended: often by *E(e)-R*, just E-R and Ee-R. Depending on the works of Routley and Meyer, and Dunn in [2, 5, 10, 11, 12], we can show the completeness for E(ec)-R. We shall also adopt the similar notation, terminology, and results found in them, and assume familiarity with them.

2. Axiom Schemes and Rules for E(ec)-R

For convenience, we present just the axiom schemes and the inference for E(ec)-R. For the remainder we shall follow the cu notation and terminology. The formation of E(ec)-R can be given following list of axiom schemes and rules:

AXIOM SCHEMES

A1. $(A \to B) \to ((B \to C) \to (A \to C))$ (suffixing)

A2. $(A \to (A \to B)) \to (A \to B)$ (contraction)

A3. $((A \to A) \to B) \to B$ (specialized assertion)

A4. $(A \land B) \to A$, $(A \land B) \to B$ ($\land$-elimination)

A5. $((A \to B) \land (A \to C)) \to (A \to (B \land C))$($\land$-introduction)

A6. $A \to (A \lor B)$, $B \to (A \lor B)$ ($\lor$-introduction)

A7. $((A \to C) \land (B \to C)) \to ((A \lor B) \to C)$ ($\lor$-elimination)

A8. $(A \land (B \lor C)) \to ((A \land B) \lor (A \land C))$(distributive law)

A9. $(\Box A \wedge \Box B) \to \Box(A \wedge B)$ where $\Box A := (A \to A) \to A$

A10. $\sim\sim A \to A$ (classical double negation)

A11. $(A \to \sim B) \to (B \to \sim A)$ (contraposition)

A12. $(A \to B) \to (A \to (A \to B))$ (expansion)

A13. $(A \to B) \vee (B \to A)$ (chain)

RULES

$A \to B, A \vdash B$ (modus ponens (MP))

$A, B \vdash A \wedge B$ (adjunction (AD))

DEFINITION

df1. $A \circ B := \sim(A \to \sim B)$

SYSTEMS

E-R.	A1 - A11;
Ee-R.	E-R + A12;
Eec-R.	Ee-R + A13.

3. Routley-Meyer frames and models for E(ec)-R

Following [2, 5, 7], calling relevant model structures *Routley-Meyer (RM) frames*, we define an *(RM) frame*. A frame is a structure $S = (U, \sqsubseteq, R, Z, ^{\bullet})$, where $(U, \sqsubseteq, R, Z)$ is a left assertional frame[2)] and $^{\bullet}$ is a unary operation on U, such that

2) That is, U is a set, Z ($\subseteq$ U) is a left lower identity ($Z \circ A \subseteq A$) satisfying the following lli
(lli) $\exists\zeta, \in Z, (R\zeta\alpha\beta)$ iff $\alpha \sqsubseteq \beta$,
$R \subseteq U3$, and $\sqsubseteq$ is a partial-order satisfying:
$R\alpha\beta\gamma$ & $\alpha' \sqsubseteq \alpha$ imply $R\alpha'\beta\gamma$,
$R\alpha\beta\gamma$ & $\beta' \sqsubseteq \beta$ imply $R\alpha\beta'\gamma$,
$R\alpha\beta\gamma$ & $\gamma' \sqsubseteq \gamma$ imply $R\alpha\beta\gamma'$.

the following definitions and postulates hold:[3] ($\zeta \in Z$)

df2. $\alpha \sqsubseteq \beta := \exists\zeta(R\zeta\alpha\beta)$

df3. $R^2\alpha\beta\gamma\delta := \exists x(R\alpha\beta x \;\&\; Rx\gamma\delta)$

df4. $R^2\alpha(\beta\gamma)\delta := \exists x(R\alpha x\delta \;\&\; R\beta\gamma x)$

df5. $S\alpha := \forall x, y(R\alpha xy \Rightarrow \exists\zeta(R\zeta xy))$

(With respect to the following postulates, just for convenience, to represent some ζ we take $\mathit{0}$, which Routley and Meyer take in their semantics. Note that $\mathit{0}$, by which we represent some ζ ($\in Z$), itself is a member of Z, i.e., $\mathit{0} \in Z$.[4])

p0. $R\alpha\beta\gamma$ and $\alpha' \sqsubseteq \alpha$ imply $R\alpha'\beta\gamma$ (monotonicity)

p1. $R^2\alpha\beta\gamma\delta \Rightarrow R^2\beta(\alpha\gamma)\delta$

p2. $R\alpha\beta\gamma \Rightarrow R^2\alpha\beta\beta\gamma$

p3. $\forall\alpha\exists x(Sx$ and $R\alpha x\alpha)$

p4. $R\alpha\alpha\alpha$ (idempotence)

p5. $R\alpha\beta\gamma \Rightarrow R\alpha\gamma^*\beta^*$

p6. $\alpha^{**} = \alpha$

p7. $R\alpha\beta\gamma \Rightarrow \exists x(Rx\beta\alpha \;\&\; R\alpha\beta\gamma)$

p8. $R\mathit{0}\alpha\beta$ or $R\mathit{0}\beta\alpha$

For E-R, df1 - df5 plus p0 - p6;

For Ee-R, definitions and postulates of E-R + p7;

More exactly to understand a left assertional frame, see [7]. Note that U is expressed as K in [5] (as well as in [11, 12]); and that, for convenience, we take a left lower identity in place of a right lower one, which Dunn and Hardegree take in [7].)

3) Note that we take df5 for the modal character of E (see [2]).

4) Often, in proofs of section 4 and 5, by $\mathit{0}$ we shall also ambiguously represent some ζ, if we do not need distinguish them, but context should determine what is intended.

For Eec-R, definitions and postulates of Ee-R + p8.

Note that each postulate for E-R, i.e., p1 to p6, is those in [2, 5, 11]; that p7 and p8 is those in [14]; that we have p0 following [5, 7]; also that $\sqsubseteq$ is a partial order (p.o.) on U with respect to E(e)-R, i.e., E-R and Ee-R, and a linear order (l.o.) on U with respect to Eec-R. Following Dunn (and Hardegree) [6] (and [7]), we regard U as a set of "states of information", and for α, $\beta \in U$, $\alpha \sqsubseteq \beta$ means that the information of α is included in that of β.

By a *model* for E(ec)-R, we mean a structure **M** = (U, $\sqsubseteq$, R, Z, *, $\models$), where (U, $\sqsubseteq$, R, Z, *) is a frame and $\models$ is a relation from U to sentences of E(ec)-R satisfying the following conditions:

(Atomic Hereditary Condition (AHC))

for a propositional variable p, if $\alpha \models p$ and $\alpha \sqsubseteq \beta$, then $\beta \models p$; (Evaluation Clauses (EC)) for formulas A, B

($\wedge$) $\alpha \models A \wedge B$ iff $\alpha \models A$ and $\alpha \models B$;

($\vee$) $\alpha \models A \vee B$ iff $\alpha \models A$ or $\alpha \models B$;

($\rightarrow$) $\alpha \models A \rightarrow B$ iff for all β, γ, if $R\alpha\beta\gamma$ and $\beta \models A$, then $\gamma \models B$;

($\sim$) $\alpha \models \sim A$ iff $\alpha^* \not\models A$.

A formula A is *true* on v at α of U just in case $\alpha \models A$; A is *verified* on **M** in case ζ (especially *0*), $\in Z$, $\models A$; A *entails* B on **M** in case $\forall x \in U$, if $x \models A$, then $x \models B$; A *E(ec)-R-entails* B just in case A entails B in every model; and A is *E(ec)-R-valid* in a frame **S** just in case it is verified in all evaluations therein.

Let Σ be the class of frames. A sentence A is E(ec)-R-valid, in symbols $\models_{E(ec)-R}$ A, if and only if $\forall S \in \Sigma$, A is E(ec)-R-valid in S.

4. Soundness for E(ec)-R

Following [2, 5], we give the soundness for E(ec)-R. To prove it, we need the Verification Lemma below. First, by an induction on A, we can easily prove

Lemma 1 (Hereditary Condition (HC) For any formula $\models$ A and $\alpha \sqsubseteq \beta$, then $\beta \models$ A}.

Since with respect to the connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, we have the same evaluations as in [2, 5, 12], we can use the Verification Lemma in them. Thus,

Lemma 2 (Verification Lemma) A entails B on v only if A $\rightarrow$ B is verified, i.e., true at ζ ($\in$ Z), on v. Thus, A entails B in a given model **M**, = (U, $\sqsubseteq$, R, Z, *, $\models$), only if A $\rightarrow$ B is E(ec)-R-valid in the model: that is, for every x ($\in$ U) if x $\models$ A then x $\models$ B only if $\zeta \models$ A $\rightarrow$ B. And A E(ec)-R-entails B only if A $\rightarrow$ B is E(ec)-R-valid.

Proof: By Lemma 2 and 3 in [12] and definitions. (Using Lemm can also prove this, see the Verification Lemma in [2, 5]). □

Let $\vdash_{E(ec)-R}$ A be the theoremhood of A in E(ec)-R. We note that each postulate was used in [2, 5, 11, 12, 14]. Thus, the Soundness for E(ec)-R is immediate.

Proposition 1 (Soundness) If $\vdash_{E(ec)\text{-}R} A$, then $\vDash_{E(ec)\text{-}R} A$.

Proof: We just prove that each instance of the axiom scheme valid in all frames, i.e., E(ec)-R-valid, as an example. For A12, it suffices by Lemma 2 to assume that $\alpha \vDash A \rightarrow B$ and show $\alpha \vDash A \rightarrow (A \rightarrow B)$. To show this last, we assume that p7 holds. First, by the assumption and $(\rightarrow)$, we can assume that $R\alpha\beta\gamma$ and $\beta \vDash A$ only if $\gamma \vDash B$. Then, by p7, there is x such that $Rx\beta\alpha$ and $R\alpha\beta\gamma$, and thus using $(\rightarrow)$, we obtain $x \vDash A \rightarrow (A \rightarrow B)$ from the assumptions. From this, we get that $\beta \vDash A$ only if $\alpha \vDash A \rightarrow B$ by $Rx\beta\alpha$, and thus by $R\alpha\beta\gamma$ $\beta \vDash A$ only if $\gamma \vDash B$, as desired.

With respect to each instance of the other axiom schemes, we note this: from Lemma 4 in [12], it follows that each instance of the conjunction, disjunction, and negation axiom schemes, i.e., A4 to A8, A10, and A11, is E(ec)-R-valid; by Lemma 5 in [11] and those of § 48.6 in [2], each instance of A1 to A3 and A9 is E(ec)-R-valid; by Proposition 1 in [14], each instance of A13 is Eec-R-valid.

From Proposition 1 in [14], it follows that each rule, i.e., preserves E(ec)-R validity. □

5. Completeness for E(ec)-R

We give the completeness for E(ec)-R by using the well-known Henkin-style proofs for modal logic, but with prime theories in place of maximal theories. To do this, we define some theories. We interpret $\vdash_{E(ec)\text{-}R}$ as the deducibility consequence relation of the logic E(ec)-R. By an *E(ec)-R-theory*, we mean a set **T** of sentences closed under deducibility, i.e., closed under MP and

AD: by a *prime E(ec)-R-theory*, a theory **T** such that if A $\vee$ B $\in$ **T**, then A $\in$ **T** or B $\in$ **T**; and by a *trivial E(ec)-R theory*, the entire set of sentences of E(ec)-R. As Dunn states in Remark 4 in [6], we note that an E(ec)-R-theory **T** contains all of the theorems of E(ec)-R. Thus it is what has been called a "regular theory" in the relevance logic literature. That is, by an E(ec)-R-theory we mean a regular E(ec)-R-theory. This means that **T** is never empty. In the results below, there is no role either for trivial E(ec)-R theories. Hence, by an "E(ec)-R theory" we mean a non-trivial one.

Let a *canonical E(ec)-R-frame* be a structure **S** = $(U_{can}, \sqsubseteq_{can}, R_{can}, Z_{can}, {}^{*}_{can})$, where $\sqsubseteq_{can}$ is an information order on U_{can}, Z_{can} is a set of any prime E(ec)-R theory, i.e., ζ_{can} ($\in Z_{can}$), $Z_{can} \subseteq U_{can}$, U_{can} is the set of prime E(ec)-R theories extending ζ_{can}, R_{can} is R below restricted to U_{can},

(1) $R\alpha\beta\gamma$ iff for any formula A, B of E(ec)-R, if A $\rightarrow$ B $\in \alpha$ and A $\in \beta$, then B $\in \gamma$,

and ${}^{*}_{can}$ is * restricted to U_{can}. We call a frame *fitting* for E(ec)-R if for each axiom scheme of E(ec)-R the corresponding semantical postulate holds. Where α is a prime theory, let α^{*} be the set of every formula A such that $\sim$A does not belong to α, i.e., $\alpha^{*} = \{A: \sim A \notin \alpha\}$.

As we mentioned above, we take the ideas of proofs from the Henkin-style completeness proofs. Thus, note that the base $\boldsymbol{0}_{can}$, i.e., $\boldsymbol{0}$, among ζ_{can} ($\in Z_{can}$), is constructed as a prime E(ec)-R-theory that excludes nontheorems of E(ec)-R, i.e., excludes A such that not $\vdash_{E(ec)\text{-}R}$ A. Note also that in proofs below, by *0*, i.e., 0_{can}, we often represent ζ_{can} (as well as *0*) if context can make clear what is intended (cf. see section 3). The partial orderedness and the linear orderedness of a canonical E(ec)-R-frame depend on * restricted on U_{can}. Then, first, it is obvious that

Proposition 2 A canonical E(ec)-R-frame is partially ordered.

Proposition 3 For E(ec)-R, a canonical frame is connected (and thus linearly ordered).

Proof: By Proposition 3 in [14]. □

Proposition 4 The canonically defined E(ec)-R-frame is a fitting for E(ec)-R}.

Proof: As an example we just prove that p7 holds. (Note that to prove the other postulates it is enough for us to point out Theorem 1 of section 48.3, and 48.6 in [2], Lemma 6 in [11], Lemma 13 in [12], and Proposition 4 in [14].)

For p7, we assume that $R\alpha\beta\gamma$. We need to show that there is a prime theory x such that $Rx\beta\alpha$ and $R\alpha\beta\gamma$. Suppose $A \rightarrow B \in \alpha$ and $A \in \beta$. Then, $A \rightarrow (A \rightarrow B) \in \alpha$ by A12. Let us take α to be x. Then, by the assumptions, we obtain $R\alpha\beta\alpha$ and $R\alpha\beta\gamma$. That is, by the assumptions $A \rightarrow (A \rightarrow B) \in \alpha$, $A \in \beta$, we get $A \rightarrow B \in \alpha$, and by the assumptions $A \rightarrow B \in \alpha$, $A \in \beta$, we obtain $B \in \gamma$, as desired. Since α is a prime theory, it ensures that there is a prime theory that satisfies p7. □

Next, we need to define an appropriate relation $\models$ on $\mathbf{S}$, = $(U_{can}, \sqsubseteq_{can}, R_{can}, Z_{can}, {}^{*}_{can})$. We define it to be that

$$a \models A \text{ iff } A \in a.$$

However, we need to verify that this satisfies AHC and EC Note that since the positive part of E(ec)-R satisfies Definiti section 42.1 in [2], we can directly use Fact 1 and Fact 2 of section 48.3 in [2], which are considered for $\mathbf{R}^+$, and thus we can use Theorem 2 of the same section.

Proposition 5 The canonically defined (U_{can}, $\sqsubseteq_{can}$, R_{can}, Z_{can}, $^{*}_{can}$, $\models$) is indeed an E(ec)-R model.

Proof: By Proposition 5 in [14]. □

Thus, (U_{can}, $\sqsubseteq_{can}$, R_{can}, Z_{can}, $^{*}_{can}$, $\models$) is an E(ec)-R model. So, since, by construction, *0* excludes our chosen nontheorem A and the canonical definition of $\models$ agrees with membership, we can state that for each nontheorem A of E(ec)-R, there is an E(ec)-R model A in which A is not $\mathit{0} \models$ A. It gives us the (weak) completeness for E(ec)-R as follows.

Theorem 1 (Weak Completeness) If $\models_{E(ec)\text{-}R}$ A, then $\vdash_{E(ec)\text{-}R}$ A.

Next, let us prove the strong completeness for E(ec)-R. As $\mathbf{R}^+$ in [2], we define A to be an *E(ec)-R consequence* of a set of formulas γ if and only if for every E(ec)-R model, whenever $a \models$ B for every B $\in \Gamma$, $a \models$ A, for (not just *0* but) all $a \in$ U. Let us say that A is *E(ec)-R deducible* from Γ if and only if A is in every E(ec)-R theory containing Γ. Then,

Proposition 6 If $\Gamma \nvdash_{E(ec)\text{-}R}$ A, then there is a prime theory that $\Gamma \subseteq \zeta$ and A $\notin \zeta$.

Proof: Take an enumeration $\{A_n : n \in \omega\}$ of the well-formed formulas of E(ec)-R. We define a sequence of sets by induction as follows:

$\zeta_0 = \{A' : \Gamma \nvdash_{E(ec)\text{-}R} A'\}$.

$\zeta_{i+1} = Th(\zeta_i \cup \{A_{i+1}\})$ if it is not the case that $\zeta_i, A_{i+1} \vdash_{E(ec)\text{-}R} A$,

ζ_i otherwise.

Let ζ be the union of all these ζ_n's. It is easy to see that theory not containing A. Also we can show that it is a prime.

Suppose toward contradiction that $B \vee C \in \zeta$ and $B, C \notin \zeta$. Then the theories obtained from $\zeta \cup B$ and $\zeta \cup C$ must both contain A. It follows that there is a conjunction of members of ζ ζ' such that $\zeta' \wedge B \vdash_{E(ec)\text{-}R} A$ and $\zeta' \wedge C \vdash_{E(ec)\text{-}R} A$. Note that if $\vdash_{E(ec)\text{-}R} A \rightarrow B$, then $A \vdash_{E(ec)\text{-}R} B$. Then, by A8 and MP, we get $(\zeta' \wedge B) \vee (\zeta' \wedge C) \vdash_{E(ec)\text{-}R} A$. And we obtain $\zeta' \wedge (B \vee C) \vdash_{E(ec)\text{-}R} A$ by the prefixing (as a theorem), A8, and MP. From this it follows that $A \in \zeta$, which is contrary to our supposition. □

Thus, by using Propositions 5, 6 we can show its strong completeness as follows.

Theorem 2 (Strong Completeness) If $\Gamma \vDash_{E(ec)\text{-}R} A$, then $\Gamma_{E(ec)\text{-}R} A$.

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