Deductions for LŁC and Its Extension

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1. Introduction

We investigated the infinite-valued logic system LŁC in [11]. We also gave the completeness of all of its normal extensions. But we did not consider deduction theorem (DT) for LŁC. We can prove it by using the well-known Herbrand-Tarski DT. However, if we have any idea to extend LŁC by adding a rule to it, we should carefully consider the use of DT (for LŁC); even though that is an extension in its characteristic algebras *(linear) quasi relative-Boolean* (qr-B) *algebras.*[1] For example, in a qr-B algebra, from a unary operation ~ that is an *involution*, i.e.,

(1) $$\sim\sim a = a;$$

(2) $$\sim(a \vee b) = \sim a \wedge \sim b,$$

we get

(2′) $$a \leq b \text{ only if } \sim b \leq \sim a,$$

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1) LŁC is the infinite(ℵ)-valued propositional system, which follows the evaluation of the Łukasiewiczs propositional calculus (ŁC) with respect to negation ~ and the Dummett's propositional calculus (LC) with respect to implication ⊃. This idea was first suggested by Rescher [7]. But he did it without enough consideration of proof theory and did not establish its completeness. We completed his idea in [11] by using its charateristic algebras (linear) qr-B algebras. qr-B algebra is the relatively pseudo-complemented lattce with quasi-complementation, which satisfies (1), (2). Exactly to understand qr-B algebras, see [11].

with which we may replace (2). (2′) implies that

(3) $$\text{if } a \Rightarrow b = 1, \text{ then } \sim b \Rightarrow \sim a = 1.$$

If we consider the "if ... , then ..." to be deducibility ⊦, then (3) may correspond to the Contraposition (CP) as follows.[2)]

(CP) $$p \supset q \vdash \sim q \supset \sim p.$$

We can prove CP just by adding the Replacement (Re) to LŁC. But to this system we can not directly apply DT for LŁC as follows.

(4) $$\vdash (p \supset q) \supset (\sim q \supset \sim p).$$[3)]

As Dunn [2] mentions "a more appropriate decision" in the use of DT for the Adjunction in the relevance logic system **R**,[4)] we also need such a decision with respect to the above example. To make clear it, we investigate some deductions for LŁC and its extension. First, we treat DT for LŁC. Second, we give DT for its extension, which may resolve the problem of this example. To do this, we use a variant of CP. We shall call it the *Disjunctive Contraposition* (DCP) and call its system LŁC_{DCP}. Thirdly, we consider natural deduction methods for LŁC and LŁC_{DCP}. Finally, we prove Re in the LŁC with CP (LŁC_{CP}) and thus show the equivalence between the LŁC with Re (LŁC_{Re}) and LŁC_{CP}. Moreover, we extend our concern on Re to LŁC_{DCP}. That is, we consider Re in an elliptical form (eRe), calling its system LŁC_{eRe}, and prove it in LŁC_{DCP}.

2) Cf. to understand this connection, see [4], chapter 6, especially 6.6. This one can be regarded as an asymmetric consequence system.

3) That is, not ⊦ (p⊃q)⊃(~q⊃~p). Some notations of the connectives of LŁC and the operators of its characteristic algebra are slightly different from the original ones in [11]. But they do not influence our investigation.

4) To understand the system **R**, see [2].

2. DT for LŁC

Before considering DT for LŁC, first, we state the axioms and the rules for LŁC, and its some theorems (see [11]).

<AXIOMS>

I.	$p\supset(q\supset p)$,
II.	$(p\supset(q\supset r))\supset((p\supset q)\supset(p\supset r))$,
III.	$(p\wedge q)\supset p$ / $(p\wedge q)\supset q$,
IV.	$((p\supset q)\wedge(p\supset r))\supset(p\supset(q\wedge r))$,
V.	$p\supset(p\vee q)$ / $q\supset(p\vee q)$,
VI.	$((p\supset r)\wedge(q\supset r))\supset((p\vee q)\supset r)$,
VII.	$p\supset(q\supset(p\wedge q))$,
VIII.	$(p\supset q)\vee(q\supset p)$,
IX.	$p\supset\sim\sim p$ / $\sim\sim p\supset p$,
X.	$\sim(p\vee q)\supset(\sim p\wedge\sim q)$ / $(\sim p\wedge\sim q)\supset\sim(p\vee q)$.

<RULES>

MP	(modus ponens),
SB	(substitution).

Without proofs we can state the following theorems:

(5)	$p\supset p$,
(6)	$(p\supset q)\supset((q\supset r)\supset(p\supset r))$,
(7)	$\sim(p\wedge q)\supset(\sim p\vee\sim q)$ / $(\sim p\vee\sim q)\supset\sim(p\wedge q)$.

For more investigation in section 3, we add tables for evaluation. An *evaluation* for LŁC is a function v: PA → [0, 1] that is extended to all well-formed formulas of $L(\sim, \supset, \wedge, \vee, p_0, p_1, \ldots)$ by the following *tables*: (PA: set of propositional variables, [0, 1]: the rationals between 1 and 0)

<TABLES>

1.	$v(\sim P)$ =	$1 - v(P)$,	
2.	$v(P \supset Q)$ =	1	if $v(P) \leq v(Q)$
		$v(Q)$	otherwise,
3.	$v(P \wedge Q)$ =	$\min(v(P), v(Q))$,	
4.	$v(P \vee Q)$ =	$\max(v(P), v(Q))$.	

By the following definition

(df1) $P \equiv Q := (P \supset Q) \wedge (Q \supset P)$,

we can express axioms IX and X as follows:

IX'. $\sim\sim p \equiv p$,

X'. $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

With respect to implication $\supset$ (including conjunction $\wedge$ and disjunction $\vee$), LŁC follows the evaluation of LC, which is an extension of the intuitionistic propositional logic **H** of Heyting. Thus, we can consider DT for LŁC by almost the same proof as **H** as well as classical logic. We can prove it using the well-known Herbrand-Tarski DT. Since Herbrand-Tarski DT is proven in standard textbooks for classical logic, we just state this theorem without proof.

Where Γ is a list of formulas of LŁC (thought of as hypothesis), we define a *deduction* from Γ to be a sequence $\Psi_1, \Psi_2, \ldots, \Psi_n$, where for each Ψ_i, $1 \leq i \leq n$, either (i) Ψ_i is in Γ, or (ii) Ψ_i is an axiom, or (iii) Ψ_i follows from preceding members of the sequence by MP. A formula φ is called to be *deducible* from Γ, in symbols $\Gamma \vdash_{LŁC} \varphi$, just in case there is some deduction from Γ ending in φ.

Lemma 2.1 *If* $\Gamma \vdash_{LŁC} \varphi$ *and* $\Gamma \vdash_{LŁC} \varphi \supset \Psi$, *then* $\Gamma \vdash_{LŁC} \Psi$.

Theorem 2.2 *(DT for LŁC) If* $\Gamma, \varphi \vdash_{LŁC} \Psi$, *then* $\Gamma \vdash_{LŁC} \varphi \supset \Psi$

3. Algebraic consideration on infinite-valuedness in $LŁC_{Re}$

The problematic situation in **R**, which we briefly mentioned in section 1, is this: in connection with the Relevant Deduction Theorem for **R**, if we apply it twice to the well-known Adjunction rule

(8) $p, q \vdash p \wedge q$,

we get

(9) $\vdash p \rightarrow (q \rightarrow (p \wedge q))$.

But as Dunn mentions in [2], this is a disastrous one, which is undesirable. As its "appropriate decision" he suggests a rule corresponding to the following axiom of Conjunction Introduction, which shows well the relevance of propositions q, r below, instead of (9).[5]

(10) $\vdash ((p \rightarrow q) \wedge (p \rightarrow r)) \rightarrow (p \rightarrow (q \wedge r))$.

With respect to the application of DT in $LŁC_{Re}$, we also need an "appropriate decision" similar to his decision (as well as those of his predecessors Anderson and Belnap [1], and Kron [5, 6]). Because our extended system $LŁC_{Re}$ has a problem similar to **R**: in $LŁC_{Re}$, we can prove CP as

5) More exactly to understand this, see [2], pp. 137-38.

follows.

Proof of CP:[6]
1. $p \supset q$ (premise),
2. $(p \wedge q) \supset p$ (III),
3. $p \supset p$ ((5)),
4. $(p \supset p) \wedge (p \supset q)$ (1, 3, VII, MP),
5. $p \supset (p \wedge q)$ (4, IV, SB, MP),
6. $p \equiv (p \wedge q)$ (2, 5, VII, MP, df1),
7. $\sim(p \wedge q) \equiv \sim p \vee \sim q$ ((7), df1),
8. $\sim p \equiv \sim(p \wedge q)$ (6, Re)
9. $\sim p \equiv \sim p \vee \sim q$ (7, 8, (6), SB, df1),
10. $\sim q \supset \sim p \vee \sim q$ (V, SB),
11. $\sim p \vee \sim q \supset \sim p$ (9, III, SB, MP),
12. $\sim q \supset \sim p$ (10, 11, (6), SB, MP).

□

But to this we can not apply DT as in (4).

However, the reason that we need a "more appropriate decision" in $L\text{Ł}C_{Re}$ is different from that in **R**. It happens because of its many-valuedness (cf. see [11]). Let us explain it by the (parts of) algebras that characterize $L\text{Ł}C_{Re}$ (as well as LŁC). (3) holds in a qr-B algebra (A, 1, $\Rightarrow$, $\sim$, $\wedge$, $\vee$), each operation $\Rightarrow$, $\sim$, $\wedge$, and $\vee$ corresponding to the connectives $\supset$, $\sim$, $\wedge$, and $\vee$, respectively[7]. But we can not state

(11) $$(a \Rightarrow b) \Rightarrow (\sim b \Rightarrow \sim a) = 1,$$

which is an algebraic expression of (4). Since whenever $\sim a < b \leq \sim b$(or $\sim b \leq b) \leq a$, (11) does not hold. That is, $a \Rightarrow b = b$ and $\sim b \Rightarrow \sim a = \sim a$. So b

6) For brevity, we assume the commutativity and in proof procedure 9 we just mention $\supset$ rules as the rules for $\equiv$ and add df1 to them.

7) Note that $\sim$, $\wedge$, and $\vee$ are used ambiguously as propositional connectives and as operators, but context should make their meaning clear.

$\Rightarrow \sim a = \sim a \neq 1$.

Following the above evaluations for ŁLC, we can represent the same situation with (11): that is, the situation that (4) does not hold. To understand it, let us consider the case that v(p) = 0.7 and v(q) = 0.5. Then $(v(p) \supset v(q)) \supset (v(\sim q) \supset v(\sim p)) = v(q) \supset v(\sim p) = v(\sim p) = 0.3 \neq 1$. Thus, semantically (4) does not hold.

4. DT for ŁLC$_{DCP}$

Whenever we apply customary DT to CP, it results in (4), which is undesirable. We can not do that in ŁLC$_{Re}$, while doing it in classical logic. The reason is its many-valuedness beyond 2-valuedness as we considered above. We need a variant of Re (or CP), which will reflect the infinite-valuedness of ŁLC.

To simplify our investigation and make clear the situation, let us consider CP rather than Re. It is clear that if we add CP to ŁLC as a rule, we meet the same situation as ŁLC$_{Re}$ with respect to the application of customary DT as in (4).[8] To resolve this problem, first, let us more consider ŁLC algebraically. We may take a clue to make "a more appropriate decision" for the ŁLC with a variant of CP from its algebra. Note that order relation $\leq$ is linearly ordered in its algebras (see [11]). So algebraically we can think of the evaluation for the implication $\supset$ as follows.

(12) $a \Rightarrow b = 1$ if $a \leq b$
$\qquad\qquad\qquad b$ otherwise, i.e., $b < a$.

Now it is important to find a case, which (a) satisfies the order relation $\leq$ and (b) makes the second in (12) to be 1 (c) under its condition. The easiest way to do it is to take the meet of a and b in (12) as the antecedent of the

8) The equivalence between these two systems will be treated in section 6.

(relative) pseudo complementation $\Rightarrow$. For example, $(a \wedge b) \Rightarrow b$, $(a \wedge b) \Rightarrow (a \Rightarrow b)$, and $(a \wedge b) \Rightarrow (a \Rightarrow (a \Rightarrow b))$ are the ones. The reason is that it makes the antecedent of $\Rightarrow$ to be less than or equal to its consequent and the whole to be 1. However, these are algebraic consideration on the sentences not only that the consequent, which constitutes the whole, can be deduced from its antecedent as a premise by using the axioms and rules for LŁC, but also that themselves can be deduced from the axioms and rules for LŁC. In place of those, let us consider this: $((a \Rightarrow b) \vee (\sim b \Rightarrow \sim a)) \Rightarrow ((\sim b \Rightarrow \sim a) \vee (a \Rightarrow b))$. The consequent of the whole seems not to be directly deduced from its antecedent as a premise by using those for LŁC, even though the whole can be deduced from them. Also, if we get

(13) $((a \Rightarrow b) \vee (\sim b \Rightarrow \sim a)) \Rightarrow ((a \Rightarrow b) \vee (\sim b \Rightarrow \sim a)) = 1$

instead of (11), we do not have any difficulty in saying that (13) holds in those algebras. So to fit our purpose let us define a new implication $\rightarrow$ as follows:

(df2) $P \rightarrow Q := (P \supset Q) \vee (\sim Q \supset \sim P)$

We call this the *disjunctive (contrapositive) implication* and the following rule the *Disjunctive Contraposition*.

(DCP) $P \rightarrow Q \vdash \sim Q \rightarrow \sim P$

To define a deduction in LŁC$_{DCP}$, we need only to add DCP to the deduction definition for LŁC in section 2 as a rule. Let (iii) in section 2 be (iii-a) and DCP be (iii-b). That is, to define a deduction of LŁC$_{DCP}$, we replace the (iii) in section 2 with (iii) that Ψ_i follows from preceding members of the sequence by (a) MP or (b) DCP. We consider the same lemma with that in section 2, but with the "LŁC$_{DCP}$" in place of the "LŁC". Then, we can obtain

Theorem 4.1 *(DT for ŁC_{DCP}) If $\Gamma, \varphi \vdash_{ŁC_{DCP}} \Psi$, then $\Gamma \vdash_{ŁC_{DCP}} \varphi \supset \Psi$.*

Proof Assume $\Gamma, \varphi \vdash_{ŁC_{DCP}} \Psi$. Then there is a deduction $\Psi_1, \Psi_2, \ldots, \Psi_m$ (Ψ is Ψ_m) from Γ, φ. We prove

(*) $$\Gamma \vdash_{ŁC_{DCP}} \varphi \supset \Psi_i$$

for i = 1, 2, ... , m. Taking i = m in (*), we have Theorem 4.1.

We prove (*) by induction on i. Thus by induction hypothesis we assume that (*) holds for all values of i that are less than some fixed value of i and prove (*) for that fixed value of i. Note, however, that the proof that i satisfies each (i), (ii), and (iii-a) is the same with that of Herbrand-Tarski DT (see [8], Theorem 4.1). We only consider the case that Ψ_i satisfies (iii-b).

Ψ_i satisfies (iii-b). Let φ be $p \rightarrow q$ and Ψ_i be $\sim q \rightarrow \sim p$.

Proof of (iii-b):

1. $\Gamma \vdash_{ŁC_{DCP}} q \supset \sim\sim q$ (IX, SB),
2. $\Gamma \vdash_{ŁC_{DCP}} (q \supset \sim\sim q) \supset (p \supset (q \supset \sim\sim q))$ (I, SB),
3. $\Gamma \vdash_{ŁC_{DCP}} p \supset (q \supset \sim\sim q)$ (1, 2, Lemma 2.1 for ŁC_{DCP}),
4. $\Gamma \vdash_{ŁC_{DCP}} (p \supset q) \supset (p \supset \sim\sim q)$ (3, II, SB, Lemma 2.1 for ŁC_{DCP}),
5. $\Gamma \vdash_{ŁC_{DCP}} \sim\sim p \supset p$ (IX),
6. $\Gamma \vdash_{ŁC_{DCP}} (\sim\sim p \supset p) \supset ((p \supset \sim\sim q) \supset (\sim\sim p \supset \sim\sim q))$ ((6), SB),
7. $\Gamma \vdash_{ŁC_{DCP}} (p \supset \sim\sim q) \supset (\sim\sim p \supset \sim\sim q)$ (5, 6, Lemma 2.1 for ŁC_{DCP}),
8. $\Gamma \vdash_{ŁC_{DCP}} (p \supset q) \supset (\sim\sim p \supset \sim\sim q)$ (4, 7, (6), SB, Lemma 2.1 for ŁC_{DCP}),
9. $\Gamma \vdash_{ŁC_{DCP}} (\sim\sim p \supset \sim\sim q) \supset ((\sim q \supset \sim p) \vee (\sim\sim p \supset \sim\sim q))$ (V, SB),
10. $\Gamma \vdash_{ŁC_{DCP}} (\sim\sim p \supset \sim\sim q) \supset (\sim q \rightarrow \sim p)$ (9, df2, SB),
11. $\Gamma \vdash_{ŁC_{DCP}} ((p \supset q) \supset (\sim\sim p \supset \sim\sim q)) \supset (((\sim\sim p \supset \sim\sim q) \supset (\sim q \rightarrow \sim p)) \supset$

((p⊃q)⊃(~q→~p))) ((6), SB),

12. Γ ⊢ $_{LŁC_{DCP}}$ (p⊃q)⊃(~q→~p) (8, 10, 11, Lemma 2.1 for LŁC$_{DCP}$),
13. Γ ⊢ $_{LŁC_{DCP}}$ (~q⊃~p)⊃((~q⊃~p)∨(~~p⊃~~q)) (V, SB),
14. Γ ⊢ $_{LŁC_{DCP}}$ (~q⊃~p)⊃(~q→~p) (13, df2, SB),
15. Γ ⊢ $_{LŁC_{DCP}}$ (((p⊃q)⊃(~q→~p))∧((~q⊃~p)⊃(~q→~p)))⊃(((p⊃q)∨(~q⊃~p))⊃(~q→~p)) (VI, SB),
16. Γ ⊢ $_{LŁC_{DCP}}$ ((p⊃q)⊃(~q→~p))∧((~q⊃~p)⊃(~q→~p)) (12, 14, VII, SB, Lemma 2.1 for LŁC$_{DCP}$),
17. Γ ⊢ $_{LŁC_{DCP}}$ ((p⊃q)∨(~q⊃~p))⊃(~q→~p) (15, 16, Lemma 2.1 for LŁC$_{DCP}$),
18. Γ ⊢ $_{LŁC_{DCP}}$ (p→q)⊃(~q→~p) (17, df2, SB).

Thus, Γ ⊢ $_{LŁC_{DCP}}$ $\varphi \supset \Psi_i$. This completes the proof of this theorem.

□

5. Natural Deduction Formations of LŁC and LŁC$_{DCP}$

To be clear our consideration on DTs, we try natural deduction formations for LŁC and LŁC$_{DCP}$. First, following customary natural deduction methods, we suggest a natural deduction system *NLŁC* for LŁC. For convenience, we just present the deductive rules including one axiom scheme for NLŁC. For the remainder we shall follow the customary notation and terminology (see [9] and [10]).

Deductions in NLŁC are generated as follows:

Basis. The single-node tree with label A is a deduction from the open assumption A; there are no closed assumptions.

Inductive Step. Let D_1, D_2 be deductions. A natural *deduction D* may be constructed according to one of the rules and the axiom scheme below. The classes $[A]^u$, $[B]^v$ below contain open assumptions of the deductions of the premises of the final inference, but are closed in the whole deduction.[9]

We express any formula by English capital letters A, B, C. For ~, ⊃, ∧, ∨, we have introduction rules (I-rules) and elimination rules (E-rules) as follows:

<INTRODUCTION RULES> <ELIMINATION RULES>

(∧I) $$\dfrac{\begin{matrix} D_1 & D_2 \\ A & B \end{matrix}}{A \wedge B}$$

(∧ER) $$\dfrac{\begin{matrix} D_1 \\ A \wedge B \end{matrix}}{A}$$ (∧EL) $$\dfrac{\begin{matrix} D_2 \\ A \wedge B \end{matrix}}{B}$$

(⊃I) $$\dfrac{\begin{matrix} [A]^u \\ D_1 \\ B \end{matrix}}{A \supset B}$$

(⊃E) $$\dfrac{\begin{matrix} D_1 & D_2 \\ A \supset B & B \end{matrix}}{B}$$

(∨IR) $$\dfrac{\begin{matrix} D_1 \\ A \end{matrix}}{A \vee B}$$ (∨IL) $$\dfrac{\begin{matrix} D_1 \\ B \end{matrix}}{A \vee B}$$

(∨E,u,v) $$\dfrac{\begin{matrix} & [A]^u & [B]^v \\ D_1 & D_2 & D_2 \\ A \vee B & C & C \end{matrix}}{C}$$

(~~I) $$\dfrac{\begin{matrix} D_1 \\ A \end{matrix}}{\sim\sim A}$$

(~~E) $$\dfrac{\begin{matrix} D_1 \\ \sim\sim A \end{matrix}}{A}$$

9) To help us to understand the "openness" and "closedness", let us consider the rule ⊃I. The above part of the line (of the rule ⊃I) means a deduction *D* with conclusion *B* and a set [A] of open assumptions, consisting of all occurrences of the formula A at top nodes of the prooftree *D* with the maker u. Its below part means that (by applying the rule ⊃I) a new prooftree is formed from *D* by adding at the bottom the conclusion A⊃B while closing the set of open assumptions A marked by u.

$$(\sim\vee I)\quad \frac{\begin{matrix}D_1\\ \sim A\wedge\sim B\end{matrix}}{\sim(A\vee B)}\qquad (\sim\vee E)\quad \frac{\begin{matrix}D_1\\ \sim(A\vee B)\end{matrix}}{\sim A\wedge\sim B}$$

The introduction and elimination rules except for those including ~ are the same as customary ones. We need just to mention the rules with respect to ~. We call ~~I and ~~E the *Double Negation Introduction* and the *Double Negation Elimination*, respectively: and ~∨I and ~∨E the *Negated Disjunction Introduction* and the *Negated Disjunction Elimination*, respectively. To these rules we add the chain axiom scheme as follows:

$$(\text{chain})\quad \frac{}{(A\supset B)\vee(A\supset B)}$$

Then, we can show that NLȽC is equivalent to (Hilbert style) axiomatic system LȽC.

Theorem 5.1 *$\Gamma\vdash_{NLȽC}\varphi$ if and only if $\Gamma\vdash_{LȽC}\varphi$.*

Proof Left to right. To show that the formation NLȽC contains (Hilbert style) formation LȽC, we deduce the axioms of LȽC in NLȽC. MP in LȽC is the same with ⊃E in NLȽC. Note that Axiom I to VII in LȽC can be proved by the above ⊃, ∧, and ∨ rules in NLȽC, since LȽC has the same axioms with **H** with respect to ⊃, ∧, and ∨, and those axioms were proved in natural deduction system of **H**. Axiom VIII is proved by the chain itself. Axiom IX can be proved by ~~I, ~~E, and ⊃I; and X by ~∨I, ~∨E, and ⊃I.

Right to left. To show that (Hilbert style) formation LȽC contains the formation NLȽC, we use the following fact that can be proved by use of MP.

(+) if $\vdash A \supset B$, then $A \vdash B$.

By applying (+) to axioms VII and III, we get $\wedge$I and $\wedge$E, respectively; by applying to V and VI, $\vee$I and $\vee$E, respectively; by applying to IX, $\sim\sim$I and $\sim\sim$E; and by applying to X, $\sim\vee$I and $\sim\vee$E. By Theorem 2.2 and MP, we get $\supset$I and $\supset$E, respectively.

□

The formation of the natural deduction system NLŁC$_{DCP}$ for LŁC$_{DCP}$ is the same as NLŁC for LŁC, except that we add to it the following rules:

($\sim_{DCP}$I) $$\frac{\begin{matrix} D_1 \\ A \to B \end{matrix}}{\sim B \to \sim A}$$ ($\sim_{DCP}$E) $$\frac{\begin{matrix} D_1 \\ \sim A \to \sim B \end{matrix}}{B \to A}$$

We call $\sim_{DCP}$I and $\sim_{DCP}$E the *Disjunctive Contrapositive Negation Introduction* and the *Disjunctive Contrapositive Negation Elimination*, respectively. Then we show that NLŁC$_{DCP}$ is equivalent to (Hilbert style) axiomatic system LŁC$_{DCP}$ as follows.

Theorem 5.2 *$\Gamma \vdash_{NLŁC_{DCP}} \varphi$ if and only if $\Gamma \vdash_{LŁC_{DCP}} \varphi$.*

Proof Left to right. To show that the formation NLŁC$_{DCP}$ contains (Hilbert style) formation LŁC$_{DCP}$, we need just to deduce DCP of LŁC$_{DCP}$ in NLŁC$_{DCP}$. DCP is proved by $\sim_{DCP}$I itself.

Right to left. To show that (Hilbert style) formation LŁC$_{DCP}$ contains the formation NLŁC$_{DCP}$, we only deduce $\sim_{DCP}$I and $\sim_{DCP}$E of NLŁC$_{DCP}$ in LŁC$_{DCP}$. $\sim_{DCP}$I is proved by DCP itself. Thus, we just prove $\sim_{DCP}$E.

Proof of $\sim_{DCP}$E in LLŁC$_{DCP}$:

1. $\sim q \rightarrow \sim p$ (premise),
2. $(\sim q \supset \sim p) \vee (\sim\sim p \supset \sim\sim q)$ (1, df2),
3. $\sim\sim q \supset q$ (IX, SB),
4. $(\sim\sim q \supset q) \supset (\sim\sim p \supset (\sim\sim q \supset q))$ (I, SB),
5. $\sim\sim p \supset (\sim\sim q \supset q)$ (3, 4, MP),
6. $(\sim\sim p \supset \sim\sim q) \supset (\sim\sim p \supset q)$ (5, II, SB, MP),
7. $p \supset \sim\sim p$ (IX),
8. $(p \supset \sim\sim p) \supset ((\sim\sim p \supset q) \supset (p \supset q))$ ((6), SB),
9. $(\sim\sim p \supset q) \supset (p \supset q)$ (7, 8, MP),
10. $(\sim\sim p \supset \sim\sim q) \supset (p \supset q)$ (6, 9, (6), SB, MP),
11. $(p \supset q) \supset ((p \supset q) \vee (\sim q \supset \sim p))$ (V, SB),
12. $(\sim q \supset \sim p) \supset ((p \supset q) \vee (\sim q \supset \sim p))$ (V, SB),
13. $(\sim\sim p \supset \sim\sim q) \supset ((p \supset q) \vee (\sim q \supset \sim p))$ (10, 11, (6), SB, MP),
14. $((\sim q \supset \sim p) \supset (\sim\sim p \supset \sim\sim q)) \supset ((p \supset q) \vee (\sim q \supset \sim p))$ (12, 13, VI, SB, MP),
15. $(p \supset q) \vee (\sim q \supset \sim p)$ (2, 14, MP),
16. $p \rightarrow q$ (15, df2)

□

6. Replacement and Contraposition: LŁC_{Re}, LŁC_{CP}, LŁC_{eRe}, and LŁC_{DCP}

We, first, prove Re in LŁC_{CP} and thus show the equivalence between LŁC_{Re} and LŁC_{CP}. Next, we extend our concern on Re to LŁC_{DCP}. That is, we consider Re in an elliptical form, which can be proved in LŁC_{DCP}.[10]

Let F(P) be any formula with some occurrences of P and F(Q) be the result of replacing one or more of those occurrences by Q. By the following rule, we mean Re:

(Re) $P \equiv Q \vdash F(P) \equiv F(Q)$.

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10) For convenience, in this section we consider the axioms for LŁC to be the axiom schemes without SB.

Then we can prove Re by using CP including the axioms and rules for LŁC. Note that disjunction $\vee$ can be defined by using $\supset$ and $\wedge$. Thus, we need only to consider the connectives $\sim$, $\supset$, $\wedge$ (see [11]). First,

Lemma 6.1 *(Re Lemma)* *In* $LŁC_{CP}$*, the following hold:*

(a) $P \equiv Q \vdash_{LŁCCP} \sim P \equiv \sim Q$;

(b) $P \equiv Q \vdash_{LŁCCP} P \supset R \equiv Q \supset R$;

(c) $P \equiv Q \vdash_{LŁCCP} R \supset P \equiv R \supset Q$;

(d) $P \equiv Q \vdash_{LŁCCP} P \wedge R \equiv Q \wedge R$;

(e) $P \equiv Q \vdash_{LŁCCP} R \wedge P \equiv R \wedge Q$.

Proof

(a)
1. $P \equiv Q$ (premise),
2. $(P \supset Q) \wedge (Q \supset P)$ (1, df1),
3. $Q \supset P$ (2, III, MP),
4. $\sim P \supset \sim Q$ (3, CP),
5. $P \supset Q$ (2, III, MP),
6. $\sim Q \supset \sim P$ (5, CP),
7. $(\sim P \supset \sim Q) \wedge (\sim Q \supset \sim P)$ (4, 6, VII, MP),
8. $\sim P \equiv \sim Q$ (7, df1).

(b)
1. $P \equiv Q$ (premise),
2. $(P \supset Q) \wedge (Q \supset P)$ (1, df1),
3. $Q \supset P$ (2, III, MP),
4. $(P \supset R) \supset (Q \supset R)$ (3, (6), MP),
5. $(Q \supset R) \supset (P \supset R)$ (similarly to 4),
6. $P \supset R \equiv Q \supset R$ (4, 5, df1).

(c)
1. $P \equiv Q$ (premise),
2. $(P \supset Q) \wedge (Q \supset P)$ (1, df1),

3. $P \supset Q$ (2, III, MP),
4. $(P \supset Q) \supset (R \supset (P \supset Q))$ (I),
5. $R \supset (P \supset Q)$ (3, 4, MP),
6. $(R \supset P) \supset (R \supset Q)$ (5, II, MP),
7. $(R \supset Q) \supset (R \supset P)$ (similarly to 6),
8. $R \supset P \equiv R \supset Q$ (6, 7, df1).

(d)
1. $P \equiv Q$ (premise),
2. $(P \supset Q) \wedge (Q \supset P)$ (1, df1),
3. $P \supset Q$ (2, III, MP),
4. $(P \wedge R) \supset P$ (III),
5. $(P \wedge R) \supset Q$ (3, 4, (6), MP),
6. $(P \wedge R) \supset R$ (III),
7. $((P \wedge R) \supset Q) \wedge ((P \wedge R) \supset R)$ (5, 6, VII, MP),
8. $(P \wedge R) \supset (Q \wedge R)$ (7, IV, MP),
9. $(Q \wedge R) \supset (P \wedge R)$ (similarly to 8),
10. $P \wedge R \equiv Q \wedge R$ (8, 9, df1).

The proof of (e) is similar to (d).

□

Let us consider three formulas P, P′, Q, where P is a subformula of Q. By Q′, we denote the formula obtained from Q by replacing any number, including zero, occurrence of P by P′. Then,

Theorem 6.2 *(Re theorem)* $P \equiv P' \vdash_{LŁCCP} Q \equiv Q'$.

Proof By induction on the complexity of Q, we prove this theorem.

Base case: Q = P. Then Q′ is P′. It is obvious.

Inductive step. If Q is not P, let us assume

(*) $P \equiv P' \vdash_{LŁCCP} K \equiv K'$.

Then one of the following holds:

(a) Q is ~K, and P is in K;
(b) Q is K⊃L, and P is in K;
(c) Q is K⊃L, and P is in L;
(d) Q is K∧L, and P is in K;
(e) Q is K∧L, and P is in L.

Case (a). By (*), $P \equiv P' \vdash_{LŁCCP} K \equiv K'$. By Lemma 6.1 (a), $K \equiv K' \vdash_{LŁCCP} \sim K \equiv \sim K'$. Thus, $P \equiv P' \vdash_{LŁCCP} \sim K \equiv \sim K'$ and so $P \equiv P' \vdash_{LŁCCP} Q \equiv Q'$. The proofs of the case (b) to (e) are similar to that of the case (a).

□

Since we proved CP in $LŁC_{Re}$ in section 3, as corollary we can state

Corollary 6.3 *$LŁC_{Re}$ and $LŁC_{CP}$ are equivalent.*

Next, for replacement theorem in $LŁC_{DCP}$, let us define

(df3) $P \leftrightarrow Q := (P \rightarrow Q) \wedge (Q \rightarrow P)$,
(df4) $P \Leftrightarrow Q := P \equiv Q$ or $P \leftrightarrow Q$.

We call ⇔ the *elliptical equivalence*[11], and thus regard the following as *the Replacement in an elliptical form* or *the elliptical Replacement*:

(eRe) $P \Leftrightarrow Q \vdash F(P) \Leftrightarrow F(Q)$.

11) We get the name "elliptical" from Dunn [3]. But (df4) is not direcrly related with the use of "elliptical" in [3]. To understand its use, see [3].

Then we can prove eRe in LŁC_{DCP}. Before doing it, we need some lemmas.

Lemma 6.4 *In LŁC_{DCP}, the following hold:*

(14) $P \equiv Q \vdash_{\text{LŁ}CDCP} P \leftrightarrow Q$;

(15) $P \equiv Q \vdash_{\text{LŁ}CDCP} \sim P \leftrightarrow \sim Q$.

Proof

Proof of (14):

1. $P \equiv Q$ (premise),
2. $(P \supset Q) \wedge (Q \supset P)$ (1, df1),
3. $P \supset Q$ (2, III, MP),
4. $(P \supset Q) \vee (\sim Q \supset \sim P)$ (3, V, MP),
5. $P \rightarrow Q$ (4, df2),
6. $Q \rightarrow P$ (similarly to 5),
7. $(P \rightarrow Q) \wedge (Q \rightarrow P)$ (5, 6, VII, MP),
8. $P \leftrightarrow Q$ (7, df3).

Proof of (15):

1. $P \equiv Q$ (premise),
2. $(P \supset Q) \wedge (Q \supset P)$ (1, df1),
3. $Q \supset P$ (2, III, MP),
4. $\sim\sim Q \supset \sim\sim P$ (3, IX, (6), MP),
5. $P \supset Q$ (2, III, MP),
6. $\sim\sim P \supset \sim\sim Q$ (5, IX, (6), MP),
7. $(\sim P \supset \sim Q) \vee (\sim\sim Q \supset \sim\sim P)$ (4, V, MP),
8. $(\sim Q \supset \sim P) \vee (\sim\sim P \supset \sim\sim Q)$ (6, V, MP),
9. $\sim P \rightarrow \sim Q$ (7, df2),
10. $\sim Q \rightarrow \sim P$ (8, df2),
11. $(\sim P \rightarrow \sim Q) \wedge (\sim Q \rightarrow \sim P)$ (9, 10, VII, MP),
12. $\sim P \leftrightarrow \sim Q$ (11, df3).

□

Lemma 6.5 *(eRe Lemma)* *In $LŁC_{DCP}$, the following hold:*

(a) $P \leftrightarrow Q \vdash_{LŁCDCP} P \leftrightarrow Q$;
(b) $P \equiv Q \vdash_{LŁCDCP} P \supset R \equiv Q \supset R$;
(c) $P \equiv Q \vdash_{LŁCDCP} R \supset P \equiv R \supset Q$;
(d) $P \equiv Q \vdash_{LŁCDCP} P \wedge R \equiv Q \wedge R$;
(e) $P \equiv Q \vdash_{LŁCDCP} R \wedge P \equiv R \wedge Q$.

Proof Since the proofs of (b) to (e) are the same with those of Lemma 6.1, we need just to prove (a).

(a)
1. $P \leftrightarrow Q$ (premise),
2. $(P \rightarrow Q) \wedge (Q \rightarrow P)$ (1, df3),
3. $P \rightarrow Q$ (2, III, MP),
4. $\sim Q \rightarrow \sim P$ (3, DCP),
5. $Q \rightarrow P$ (2, III, MP),
6. $\sim P \rightarrow \sim Q$ (5, DCP),
7. $(\sim P \rightarrow \sim Q) \wedge (\sim Q \rightarrow \sim P)$ (4, 6, VII, MP),
8. $\sim P \leftrightarrow \sim Q$ (7, df3).

□

From Lemma 6.4 (14) and Lemma 6.5, we get

Corollary 6.6 *In $LŁC_{DCP}$, the following hold:*

(16) $P \equiv Q \vdash_{LŁCDCP} P \supset R \leftrightarrow Q \supset R$;
(17) $P \equiv Q \vdash_{LŁCDCP} R \supset P \leftrightarrow R \supset Q$;
(18) $P \equiv Q \vdash_{LŁCDCP} P \wedge R \leftrightarrow Q \wedge R$;
(19) $P \equiv Q \vdash_{LŁCDCP} R \wedge P \leftrightarrow R \wedge Q$.

Let us consider the formulas P, P′, Q, and Q′ as stated above. Then,

Theorem 6.7 *(eRe theorem)* $P \Leftrightarrow P' \vdash_{L\text{Ł}CDCP} Q \Leftrightarrow Q'$.

Proof By induction on the complexity of Q, we prove this theorem.

Base case: Q = P. Then Q′ is P′. It is obvious.

Inductive step. If Q is not P, let us assume

$$(*) \qquad P \Leftrightarrow P' \vdash_{L\text{Ł}CDCP} K \Leftrightarrow K'.$$

Then one of the following holds:

(a) Q is ~K, and P is in K;
(b) Q is K⊃L, and P is in K;
(c) Q is K⊃L, and P is in L;
(d) Q is K∧L, and P is in K;
(e) Q is K∧L, and P is in L.

With the reason similar to Lemma 6.5, we only prove (a). (Note, however, that with respect to (b) to (e), we should also consider the case that $P \Leftrightarrow P' \vdash_{L\text{Ł}CDCP} Q \Leftrightarrow Q'$, which can be proved by Corollary 6.6 (16) to (19).)

Case (a). By (*), $P \Leftrightarrow P' \vdash_{L\text{Ł}CDCP} K \Leftrightarrow K'$ and $P \Leftrightarrow P' \vdash_{L\text{Ł}CDCP} K \Leftrightarrow K'$ is one of (i) $P \equiv P' \vdash_{L\text{Ł}CDCP} K \equiv K'$, (ii) $P \equiv P' \vdash_{L\text{Ł}CDCP} K \leftrightarrow K'$, and (iii) $P \leftrightarrow P \vdash_{L\text{Ł}CDCP} K \leftrightarrow K$. (Note that it is not the case that $P \leftrightarrow P' \vdash_{L\text{Ł}CDCP} K \equiv K$.)

Subcase (a-i). Let $P \equiv P' \vdash_{L\text{Ł}CDCP} K \equiv K'$. By Lemma 6.4 (15), $K \equiv K' \vdash_{L\text{Ł}CDCP} {\sim}K \leftrightarrow {\sim}K'$. Thus, $P \equiv P' \vdash_{L\text{Ł}CDCP} {\sim}K \leftrightarrow {\sim}K'$ and so $P \equiv P' \vdash_{L\text{Ł}CDCP} Q \leftrightarrow Q'$.
Subcase (a-ii). Let $P \equiv P' \vdash_{L\text{Ł}CDCP} K \leftrightarrow K'$. By Lemma 6.5 (a), $K \leftrightarrow K' \vdash_{L\text{Ł}CDCP} {\sim}K \leftrightarrow {\sim}K'$. Thus, $P \equiv P' \vdash_{L\text{Ł}CDCP} {\sim}K \leftrightarrow {\sim}K'$ and so $P \equiv P'$

$\vdash_{LŁC_{DCP}} Q \leftrightarrow Q'$.

Subcase (a-iii). Let $P \leftrightarrow P' \vdash_{LŁC_{DCP}} K \leftrightarrow K'$. By Lemma 6.5 (a), $K \leftrightarrow K' \vdash_{LŁC_{DCP}} \sim K \leftrightarrow \sim K'$. Thus, $P \leftrightarrow P' \vdash_{LŁC_{DCP}} \sim K \leftrightarrow \sim K'$ and so $P \leftrightarrow P' \vdash_{LŁC_{DCP}} Q \leftrightarrow Q'$.

□

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